MASTERS PROJECT

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Gross-Stark conjecture

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Preface

For a number field K, let $\zeta_K(s)$ denote the Dedekind zeta function, a priori defined only for $\mathrm{Re}(s) > 1$ by the Euler product

$$\prod_{\mathfrak{p}:\text{finite places}} \left(1 - \frac{1}{\mathbb{N}\mathfrak{p}^s}\right)^{-1}$$

We can analytically continue this function to the entire complex plane and obtain a functional equation as well. Dirichlet was able to show that there is a pole of $\zeta_K(s)$ at $s = 1$ and infact the residue at $s = 1$ is of utmost importance. He showed that

$$\mathrm{Res}_{s=1}\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}}\frac{hR}{e}$$

where r_1 is the number of real embeddings, r_2 the number of complex embeddings, d is the absolute discriminant, R is the regulator, h is the class number, e is the number of roots of unity contained in K. This is one of the many instances where the special value of a L-function is related to an arithmetic invariant of the underlying algebraic object.

Artin introduced L-functions $L(\chi, s)$ attached to any complex representation $\chi : \mathrm{Gal}(\overline{K}/K) \to \overline{\mathbb{Q}}^{\times}$ of the absolute Galois group of number fields. In a series of papers starting in [Sta71], Stark studied the special values of these L-functions and conjectured that

$$\mathrm{Res}_{s=0}\frac{L(\chi, s)}{s^{r_\chi}} = R(\chi)A(\chi)$$

where r_χ is given, $R(\chi)$ is the generalised regulator and $A(\chi)$ is some arithmetic constant. Stark's conjecture was refined and reformulated by Tate in [Tat84]. Soon after, Deligne-Ribet [DR80], Cassou-Nogues[Cas79], Barsky[Bar78] were able to construct p-adic L-functions which interpolate to special values of these L-functions.

Gross conjectured a similar formula for the leading term of the p-dic L-functions

$$\mathrm{Res}_{s=0}\frac{L_p(\omega\chi, s)}{s^r} = R_p(\chi)A(\chi)$$

where R_p is the p-adic regulator. This conjecture is known as the Gross-Stark conjecture. Gross proved the $K = \mathbb{Q}$ case and using the methods developed in [Wil88][Wil90], Dasgupta-Darmon-Pollack [DDP11] were able to prove the conjecture for the rank one case under the additional hypothesis that Leopoldt's conjecture holds. The assumption on Leopoldt's conjecture was removed by Ventullo in [Ven15][Ven14] and the Gross-Stark conjecture was proved in full generality by Dasgupta-Kakde-Ventullo in [DKV18]. My masters thesis is to understand the proof of the Gross-Stark conjecture in the two seminal papers.

The current version of the thesis contains an exposition to the work of Dasgupta-Darmon-Pollack. In the first chapter, we give an introduction to Stark's conjecture as formulated by Tate in his highly regarded book [Tat84] (the Bible for Stark's conjectures) and Gross's p-adic formulation of the conjecture.

Chapter 2 contains the cohomological interpretation of the Gross-Stark conjecture and reduces the conjecture to finding a cohomology class of the appropriate type.

Chapter $3, 4$ deal with the contruction of the cohomology class using the methods initiated in Wiles papers [Wil86][Wil88][Wil90] (they are in turn inspired by [Rib90]).

The current draft is unfinished. We intend to cover the work of Dasgupta-Kakde-Ventullo in the final draft and also provide an excursion into other beautiful conjectures of Stark and their refinements. Some proofs have also been excluded to make the exposition brief and avoid repeating what has already been written beautifully in the original papers. We only provide the details that we felt were missing while reading the papers.

Notation

Let k be a global field, i.e. a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$. The places or equivalence classes of absolute values of k is denoted by $v, v', \ldots$. If $\mathbb{Q} \subseteq k$, we use $\mathfrak{p}, \mathfrak{q}, \ldots$ to denote the finite places of k to distinguish it from other ideals of the ring of integers of k (denoted by other fractal letters). Given a finite extension K/k, by $w, w', \ldots$ we denote the places of K that extend $v, v', \ldots$. We use capital gothic letters $\mathfrak{P}, \mathfrak{Q}$ to denote the places of K that divide $\mathfrak{p}, \mathfrak{q}$.

The complete local fields are denoted by $k_v, K_w, k_\mathfrak{p}, K_\mathfrak{P}$; the ring of integers by $\mathcal{O}_v, \mathcal{O}_w, \mathcal{O}_\mathfrak{p}, \mathcal{O}_\mathfrak{P}$. If w is a place of K extending v, the degree of extension $[K_w : k_v]$ is denoted by $[w : v]$.

If S is a finite set of places of k containing all the Archimedean places of k, we can define the ring of S-integers

$$\mathcal{O}_S := \{x \in k : x \in \mathcal{O}_\mathfrak{p} \ \forall\ \mathfrak{p} \notin S\} = \bigcap_{\mathfrak{p} \notin S} \mathcal{O}_\mathfrak{p}$$

to be the Dedekind domain obtained by inverting all the primes of k contained in S.

We simply write $||_v, ||_w, ||_\mathfrak{p}, ||_\mathfrak{P}, \ldots$ for the normalised absolute values attached to the places indicated in the subscript. If $x \in k^\times$, we have $\mu(xU) = |x|_v \mu(U)$ for all compact sets U in the interior of k_v and all choices of Haar measure μ on the additive group k_v. More explicitly, the absolute values are

$$|x|_v = \begin{cases} \text{usual absolute value} & \text{if } k_v \simeq \mathbb{R} \\ \text{sqaure of usual absolute value} & \text{if } k_v \simeq \mathbb{C} \\ \mathbb{N}v^{-1} & \text{if } k_v \text{ is non-Archimedean} \end{cases}$$

For $x \in \mathbb{Z}_p^\times$, we have the factorisation

$$\begin{aligned} \mathbb{Z}_p^\times &= (\mathbb{Z}/2p\mathbb{Z})^\times \times (1 + 2p\mathbb{Z}_p) \\ &= \omega(x)\langle x \rangle \end{aligned}$$

with ω and $\langle\cdot\rangle$ defined by the decomposition above.

$\langle\cdot\rangle$ will also be used to denote the ideal generated by $\cdot$. The two usage shall be clear from context.

1. Introduction

This chapter (more specifically §§1.1-1.5) follows [Tat84, §1 , §§0-4] closely. I have provided proof of statements that the book chooses to leave. I claim no originality in the presentation. This chapter(§§1.1-1.5) is mostly a translation of the chapter in *loc. cit.*

1.1. Dirichlet's analytic class number formula

Suppose k is a number field (finite extension of $\mathbb{Q}$), and $\zeta_k(s)$ is the Dedekind zeta function of k, defined for $\mathrm{Re}(s) > 1$ by the Euler product

$$\zeta_k(s) := \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathbb{N}\mathfrak{p}^s}\right)^{-1} \tag{1.1}$$

where the product is over all the prime ideals of k. A famous theorem of Dedekind [Theorem 40 Mar18, p. 123], a generalisation of a theorem of Dirichlet, states that

Theorem 1. *$\zeta_k(s)$ has a simple pole at $s = 1$, and the residue at $s = 1$ is*

$$\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}} \frac{hR}{e} \tag{1.2}$$

where r_1 (resp. r_2) is the number of real (resp. complex) embeddings of K, d the discriminant of k, h the class number of k, and e the number of roots of unity contained in k.

The functional equation of $\zeta_k(s)$ (cf. Appendix F) allows us to rewrite this theorem into a statement around the point $s = 0$.

Proposition 2. *The Taylor expansion of $\zeta_k(s)$ around $s = 0$ is given by*

$$\zeta_k(s) = -\frac{hR}{e} s^{r_1+r_2-1} + O(s^{r_1+r_2}) \tag{1.3}$$

Proof. If $\Lambda_k(s) = 2^{r_2(1-s)}|d|^{s/2}\pi^{-ns/2}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\zeta_K(s)$, then by the functional equation we have

$$\Lambda(s) = \Lambda(1-s)$$

as $W(\chi) = 1$ if χ is trivial. Thus, using Dirichlet's analytic class number formula at $s = 1$, and the fact that $\Gamma(s)$ has a pole at $s = 0$ with residue 1, we have

$$s\zeta_K(s) \sim -\frac{hR}{e}s^{r_1+r_2}$$

as s goes to 0. This completes the proof. □

The above proposition gives us the first non-zero term in the Taylor series expansion of $\zeta_k(s)$ around $s = 0$. Stark's conjecture will state a similar result but for Artin L-functions. Before proceeding further, we will state Dirichlet's analytic class number formula in a slightly general setting of S-units.

Let S be a finite set of places of k containing the Archimedean places S_∞. For $\mathrm{Re}(s) > 1$, we can define the generalised zeta function

$$\zeta_{k,S}(s) = \prod_{\mathfrak{p}\notin S}\left(1 - \frac{1}{\mathbb{N}\mathfrak{p}^s}\right)^{-1} \tag{1.4}$$

By $Cl(\mathcal{O}_{k,S})$ we will denote the ideal class group of the S-integers, and $h_{k,S}$ will denote the size of this class group.

Definition 3. *$\mathcal{O}_S^\times$ is finitely generated abelian group and thus has a free-part and a torsion part. By the S-unit theorem (cf. the next section) the rank of the free part is $r = |S| - 1$. Let $\{u_1, \dots, u_r\}$ be a set of fundamental units modulo the torsion $(\mathcal{O}_S^\times)_{\mathrm{tors}}$. The regulator R_S is defined to be*

$$R_S = \left| \det_{\substack{1\le i\le r\\ v\in S\setminus\{v_0\}}} (\log|u_i|_v) \right| \tag{1.5}$$

where v_0 is an arbitrarily chosen Archimedean prime in S.

Remark 4. *A priori, it looks like the definition depends on the choice of the Archimedean place v_0, and the choice of basis $\{u_1, \dots, u_r\}$. But, the dependence on v_0 can be removed by the product formula. Let $\{\epsilon_1, \dots, \epsilon_r\}$ be another set of* *fundamental units. Then, one can show that*

$$R_S(\{u_i\}) = R_S(\{\epsilon_i\})[\langle u_i\rangle : \langle \epsilon_i\rangle]$$

where the index is just the determinant of the transformation matrix. For fundamental units, the determinant is 1 *and thus* R_S *does not depend on the choice of fundamental units.*

Lemma 5. *Let* $\mathfrak{p}$ *be a place of* k *not contained in* S*. Let* $T = S \cup \{\mathfrak{p}\}$*. Let* m *be the order of* $\mathfrak{p}$ *in the ideal class group of* S*-integers* $\mathcal{O}_S$*. We can conclude that*

1. $h_S = mh_T$
2. $R_T = m(\log \mathbb{N}\mathfrak{p})R_S$
3. $\zeta_{k,T}(s) \sim (\log \mathbb{N}\mathfrak{p})s\zeta_{k,S}(s)$ *in the neighbourhood of* $s = 0$

Proof. There is a natural map from $I(\mathcal{O}_S) \to I(\mathcal{O}_T)$ via $\mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_T$. This map is surjective (look at the prime factorisation). Now, combining with the standard projection map $I(\mathcal{O}_T) \to C(\mathcal{O}_T)$ we have a surjective map $C(\mathcal{O}_S) \to C(\mathcal{O}_T)$. To finish the proof of first assertion, it is enough to show the following sequence is exact:

$$0 \longrightarrow [\mathfrak{p}] \longrightarrow C(\mathcal{O}_S) \longrightarrow C(\mathcal{O}_T) \longrightarrow 0$$

where $[\mathfrak{p}]$ is the class of $\mathfrak{p}$ in the ideal class group $C(\mathcal{O}_S)$. We have already shown the surjection. Let us prove the injectivity of the map $[\mathfrak{p}] \to C(\mathcal{O}_S)$. Let $\mathfrak{a} \in I(\mathcal{O}_S)$ be in the kernel of the map $C(\mathcal{O}_S) \to C(\mathcal{O}_T)$, then there exists $\alpha \in k^\times$ such that $\mathfrak{a}\mathcal{O}_T = \alpha\mathcal{O}_T$. As $S \subseteq T$, we can conclude that $v_\mathfrak{q}(\mathfrak{a}) = v_\mathfrak{q}(\alpha\mathcal{O}_S)$ for all places $\mathfrak{q} \neq \mathfrak{p}$. Thus, $\mathfrak{a} = \mathfrak{p}^e\alpha\mathcal{O}_S$ with $e = v_\mathfrak{p}(\mathfrak{a}) - v_\mathfrak{p}(\alpha)$. As both sides are fractional ideals of $\mathcal{O}_S$ with same valuation at all places, this completes the proof of first assertion.

Let $\{u_1, \ldots, u_r\}$ be a set of fundamental units of $\mathcal{O}_S^\times/(\mathcal{O}_S)_{tors}$. If $\mathfrak{p}^m = \varpi\mathcal{O}_S$, then $\{u_1, \ldots, u_r, \varpi\}$ is a system of units for $\mathcal{O}_T^\times/(\mathcal{O}_T)_{tors}$. Indeed, if $u \in \mathcal{O}_T^\times$, then after scaling with appropriate power of ϖ we can assume that $0 \leq v_\mathfrak{p}(u) \leq m - 1$. Then, $u\mathcal{O}_S = \mathfrak{p}^{v_\mathfrak{p}(u)}$ as the valuations of both sides is equal for all places. But the order of $[\mathfrak{p}]$ in $C(\mathcal{O}_S)$ is m and so $v_\mathfrak{p}(u) = 0$ or equivalently, $u \in \mathcal{O}_S^\times$. Back to the second assertion. Note that $v_\mathfrak{q}(\varpi) = 0$ for all $\mathfrak{q} \neq \mathfrak{p}$ and so the matrix M_T corresponding to the regulator R_T has the form

$$M_T = \left[\begin{array}{c|c} M_S & \star \\ \hline \mathbf{0} & \log|\varpi|_{\mathfrak{p}} \end{array}\right]$$

Hence, $R_T = R_S \log|\varpi|_{\mathfrak{p}} = R_S \cdot m \cdot \log \mathbb{N}\mathfrak{p}$.

The third assertion follows from the observation

$$\zeta_{k,T}(s) = \left(1 - \mathbb{N}\mathfrak{p}^{-s}\right) \zeta_{k,S}(s)$$

and taking limit as $s \to 0$. □

The following theorem follows immediately from the above lemma.

Theorem 6. *In the neighbourhood of $s = 0$, we have*

$$\zeta_{k,S}(s) \sim -\frac{h_S R_S}{e} s^{\#S-1}$$

1.2. Artin L-functions

Suppose K is now a finite Galois extension of k, with Galois group G. One has

$$\chi : G \to \mathbb{C}$$

a character of a representation $G \to \mathrm{GL}(V)$ with V a finite dimensional vector space over $\mathbb{C}$.

Fix a finite set of places of k, S, then one can simply write

$$L_S(s,\chi) = \prod_{\mathfrak{p} \notin S} \det(1 - \sigma_{\mathfrak{P}} N\mathfrak{p}^{-s} \mid V^{I_{\mathfrak{P}}})^{-1}$$

for the Artin L-function (relative to S) attached to χ. Here $\mathfrak{P}$ denotes an arbitrary place of K lying above $\mathfrak{p}$, and $\sigma_{\mathfrak{P}} \in G_{\mathfrak{P}}/I_{\mathfrak{P}}$ is the Frobenius automorphism of the extension of the residue fields $\mathfrak{P}/\mathfrak{p}$. The function $L(s,\chi)$ does not depend on the choice of the prime $\mathfrak{P}$ as all the Frobenius elements are conjugate to each other and determinant is invariant under change of basis.

In a neighbourhood of $s = 0$, set

$$L_S(s,\chi) = c(\chi)s^{r(\chi)} + \mathcal{O}(s^{r(\chi)+1})$$

We are interested in finding $c(\chi)$ but first we will determine the multiplicity $r(\chi)$. Let S_K be the finite set of places of K lying above the places in S, the finite set of places of k; and Y the free abelian group with basis S_K. Let

$$X = \left\{ \sum_{w\in S_K} n_w w \in Y : \sum_{w\in S_K} n_w = 0 \right\}$$

The Galois group G acts naturally by permutation of the places w dividing v for each $v \in S$. Thus we obtain a G-module structure on Y and on X. We have an exact sequence of G-modules :

$$0 \longrightarrow X \longrightarrow Y \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$\sum n_w w \longmapsto \sum n_w$$

Definition 7 (Notation)**.** *For a $\mathbb{Z}$-module B and a subring A of $\mathbb{C}$, by AB we mean the tensor product $A \otimes_{\mathbb{Z}} B$. Let χ_X be the character of the representation $\mathbb{C}X$ of G, and similarly χ_Y of $\mathbb{C}Y$.*

Remark 8. *Note that $\chi_X = \chi_Y - 1$.*

Evidently, $\chi_Y = \bigoplus_{v\in S} \mathrm{Ind}_{G_w}^{G} \mathbf{1}_{G_w}$, where for each $v \in S$, w is a place of K dividing v chosen arbitrarily. In particular, χ_Y and χ_X take their values in $\mathbb{Z}$.

Proposition 9. *If χ is a character of a $\mathbb{C}[G]$-module V (finite dimensional $\mathbb{C}$ vector space), then*

$$r(\chi) = \left(\sum_{v\in S} \dim V^{G_w} \right) - \dim V^G = \langle \chi, \chi_X \rangle_G = \dim_{\mathbb{C}} \mathrm{Hom}_G(V^*, \mathbb{C}X)$$

where V^* is the dual of V.

Proof. We have a canonical homomorphism $\mathrm{Hom}_{\mathbb{C}}(V^*, \mathbb{C}X) \simeq V^{**} \otimes_{\mathbb{C}} \mathbb{C}X \simeq V \otimes_{\mathbb{C}} \mathbb{C}X$. Thus, $\mathrm{Hom}_G(V^*, \mathbb{C}X) \simeq (V \otimes_{\mathbb{C}} \mathbb{C}X)^G$. Using the othogonality of characters one has $\dim_{\mathbb{C}} \mathrm{Hom}_G(V^*, \mathbb{C}X) = \langle \chi\chi_X, \mathbf{1} \rangle_G = \langle \chi, \overline{\chi}_X \rangle_G$. Moreover, $\chi_X = \overline{\chi}_X$ (χ_X only takes integer values) and thus we have the last equality.

The second equality follows from Frobenius reciprocity in the following way:

$$\begin{aligned}\langle \chi, \chi_X \rangle_G &= \langle \chi, \chi_Y \rangle_G - \langle \chi, 1 \rangle_G \\ &= \sum_{v \in S} \langle \chi, \mathrm{Ind}_{G_w}^G \mathbf{1}_{G_w} \rangle_G - \langle \chi, 1 \rangle_G \\ &= \sum_{v \in S} \langle \chi|_{G_w}, \mathbf{1}_{G_w} \rangle_{G_w} - \dim_{\mathbb{C}} V^G \\ &= \sum_{v \in S} \dim_{\mathbb{C}} V^{G_w} - \dim_{\mathbb{C}} V^G\end{aligned}$$

It remains to show the first equality. By Brauer-Nesbitt theorem,

$$\chi = \sum_{\psi} n_\psi \mathrm{Ind}_H^G \psi$$

where ψ are 1 dimensional characters of subgroups H of G. Again, by Frobenius reciprocity

$$\langle \chi, \chi_X \rangle_G = \sum n_\psi \langle \chi|_H, \psi \rangle_H$$

Next, by properties of L-functions

$$r(\chi) = \sum n_\psi r(\psi)$$

Comparing the two relations tell us that it is sufficient to study just the 1 dimensional characters ψ.

If $\chi = \mathbf{1}_G$, then $L_S(s, \chi) = \zeta_{k,S}(s)$ and so using Theorem 6 gives us

$$r(\chi) = \#S - 1 = \left(\sum_{v \in S} \dim V^{G_w} \right) - \dim V^G$$

If χ is a 1-dimensional character but not the trivial character, then $V^G = \{0\}$. This handles one summand. The other summand is a bit tricky. Recall the functional equation of $L_S(s, \chi)$

$$\Lambda(1 - s, \chi) = W(\chi) \Lambda(s, \overline{\chi}) \tag{1.6}$$

with

$$\Lambda(s, \chi) = \Gamma_{\mathbb{R}}(s)^{a_1} \Gamma_{\mathbb{R}} \left(\frac{s+1}{2} \right)^{a_2} L(s, \chi) \Gamma_{\mathbb{C}}(s)^{r_2} \tag{1.7}$$

and

$$a_1 = \sum_{v \text{ real}} \dim V^{G_w}, a_2 = \sum_{v \text{ real}} \text{codim} V^{G_w} \tag{1.8}$$

It is a well known fact that $L(s,\chi)$ does not vanish at $s=1$ and $W(\chi)$ is a non-vanishing holomorphic function. So, if we compare order of vanishing on both sides of the functional equation, we get

$$-a_1 - r_2 + r_{S_\infty} = 0 \Leftrightarrow r_{S_\infty} = a_1 + r_2 = \sum_{v|\infty} \dim V^{G_w}$$

where the last equality comes from the fact that $\dim_{\mathbb{C}} V = 1$ and r_2 is the number of complex embeddings of k in $\overline{\mathbb{Q}}$. As

$$L_S(s,\chi) = \prod_{\substack{\mathfrak{p}\in S\setminus S_\infty \\ \chi(I_\mathfrak{p})=1}} \left(1 - \chi(\mathfrak{p})\mathbb{N}\mathfrak{p}^{-s}\right) L_{S_\infty}(s,\chi)$$

As $G_\mathfrak{p}$ is generated by $I_\mathfrak{p}$ and a Frobenius $\sigma_\mathfrak{p}$, the order of vanishing of $L_S(s,\chi)$ is exactly

$$\begin{aligned} r_S(\chi) &= \#\{\mathfrak{p} \in S\setminus S_\infty : \chi(G_\mathfrak{p}) = 1\} + r_{S_\infty} \\ &= \sum_{\mathfrak{p}\in S\setminus S_\infty} \dim V^{G_\mathfrak{p}} + r_{S_\infty} \\ &= \sum_{\mathfrak{p}\in S} \dim V^{G_\mathfrak{p}} \end{aligned}$$

This completes the proof. □

We will record the observation made in the proof as it is very crucial for our purposes.

Theorem 10. *If χ is a 1-dimensional character of G, then*

$$r_S(\chi) = \begin{cases} \#S - 1 & \text{if } \chi = \mathbf{1}_G \\ \#\{v \in S : \chi(G_v) = 1\} & \text{otherwise} \end{cases}$$

1.3. Stark's regulator

We will now introduce the type of regulator attached to χ which will figure in the principal conjecture of Stark. Denote by

$$U = \{x \in K^\times : |x|_w = 1 \; \forall \; w \notin S_K\}$$

the group of S_K-units of K, and consider the logarithmic embedding

$$\begin{aligned} \lambda : U &\longrightarrow \mathbb{R}X \\ u &\mapsto \sum_{w \in S_K} \log |u|_w w \end{aligned}$$

where X is as defined in §1.2. This is used in the proof of the theorem of S-units ([Wei95, IV–4, Theorem 9]). The kernel is the group $\mu(K)$ of roots of unity contained in K, and the image is a lattice in $\mathbb{R}X$. We shall record this as a theorem as it will be cited often.

Theorem 11 (Dirichlet S-unit theorem)**.** *The kernel of λ is the group of roots of unity $\mu(K)$ contained in K, and the image is a full lattice in $\mathbb{R}X$ with rank $\#S-1$. Hence, the group $U/\mu(K)$ is a free abelian group on the $\#S-1$ generators and $1 \otimes \lambda : \mathbb{R}U \to \mathbb{R}X$ is an isomorphism.*

On tensoring with $\mathbb{C}$, λ induces isomorphism (again called λ):

$$\mathbb{C}U \xrightarrow{\sim} \mathbb{C}X$$

compatible with the natural action of G on U and X.

This implies that the two representations of G $\mathbb{Q}U$ and $\mathbb{Q}X$ are isomorphic over $\mathbb{Q}$ (Recall that we showed the invariance of this isomorphy of finite group representations by extension of scalars (in characteristic zero) either by passing to the associated characters [Ser77, §12.1] , the note after prop. 33 or by characterising an isomorphism as a homomorphism with non-zero determinant-refer to [CF10, p. 110]).

Therefore,

$$f : \mathbb{Q}X \xrightarrow{\sim} \mathbb{Q}U \tag{1.9}$$

is an isomorphism of $\mathbb{Q}G$-module, and note again

$$f\colon \mathbb{C}X \xrightarrow{\sim} \mathbb{C}U$$

its complexification.

The automorphism $\lambda \circ f$ of $\mathbb{C}X$ induces an automorphism (functorial)

$$\mathrm{Hom}_G(V^*, \mathbb{C}X) \xrightarrow{(\lambda\circ f)_V} \mathrm{Hom}_G(V^*, \mathbb{C}X)$$

$$\varphi \longmapsto \lambda \circ f \circ \varphi$$

Recall that V^* is the dual of the vector space V and following Theorem 10, the dimension of $\mathrm{Hom}_G(V^*, \mathbb{C}X)$ is exactly $r(\chi)$.

Definition 12. *The Stark regulator attached to f is defined as:*

$$R(\chi, f) = \det((\lambda \circ f)_V) \tag{1.10}$$

It is evident that $R(\chi, f)$ does not depend on the choice of the vector space V of χ. The choice of f, on the contrary, is not negligible.

1.4. Stark's principal conjecture

In the notations in the previous two paragraphs, the statement of the conjecture is as follows:

Conjecture 13. *Let $A(\chi, f) = R(\chi, f)/c(\chi) \in \mathbb{C} \in \mathbb{C}$. Then, for all automorphisms σ of $\mathbb{C}$, one has the relation*

$$A(\chi, f)^{\alpha} = A(\chi^{\alpha}, f)$$

where $\chi^{\alpha} = \alpha \circ \chi\colon G \to \mathbb{C}$.

We can decompose our statement in the following manner :

1. $A(\chi, f)$ belongs to $\mathbb{Q}(\chi)$

2. For all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, $A(\chi, f)^{\sigma} = A(\chi^{\sigma}, f)$

Here, $\mathbb{Q}(\chi)$ is the field of values of χ. It is a cyclotomic extension, and thus Galois extension of $\mathbb{Q}$. ([Ser77, §2.1]).

It seems appropriate to reformulate the conjecture starting from the situation relative to an E (coefficient field) which allows embeddings in $\mathbb{C}$. It is, in fact sufficient to consider only number fields (finite extension of $\mathbb{Q}$).

Suppose E is a field of characteristic 0 and $\chi : G \to E$ a character of the representation $G \to \mathrm{GL}_E(V)$, where V is a vector space of finite dimension over E. (Recall that G is the Galois group of the extension K/k). Instead of assuming f is rational (as in in the previous section), let us take any G-homomorphism $f : X \to EU$.

For all $\alpha \in \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$, one can deduce from χ and V a complex character $\chi^{\alpha} = \alpha \circ \chi$ of G and its complexification $V^{\alpha} = V \otimes_{E,\alpha} \mathbb{C}$, to which 2.3 applies. In particular for each α, we can associate a L-function $L(s, \chi^{\alpha})$. Moreover, $f^{\alpha} : \mathbb{C} \to \mathbb{C}U$ is defined by $\mathbb{C}$-linearity from $(\alpha \circ 1) \circ f : X \to \mathbb{C}U$, and induces the endomorphism $(\lambda \circ f^{\alpha})_{V^{\alpha}}$ of $\mathrm{Hom}_G(V^{\alpha *}, \mathbb{C}X)$. Denote by $R(\chi^{\alpha}, f^{\alpha})$ its determinant (it is independent of the vector space V over E associated to χ).

In this context, we are then led to the

Conjecture 14. *There exists an element $A(\chi, f)$ of E such that, for all $\alpha : E \to \mathbb{C}$, we have*

$$R(\chi^{\alpha}, f^{\alpha}) = A(\chi, f)^{\alpha} \cdot c(\chi^{\alpha}) \tag{1.11}$$

Remark: The complex conjugation being continuous, it is easy to see that $\overline{A(\chi, f)} = A(\overline{\chi}, f)$.

1.4.1. Changing the isomorphism f

Proposition 15. *The conjecture* 13 *implies conjecture* 14*.*

It is clear that one can always, in Conjecture 14, one can reduce to the case $E = \mathbb{C}$ and fix an arbitrary embedding $\alpha : E \to \mathbb{C}$. It is sufficient to show the independence of choice of f in this case to show that Conj. 13 implies Conj. 14 :

If the statement in Conj. 14, with $E = \mathbb{C}$, is true for a particular choice of isomorphism $f_0 : \mathbb{C}X \xrightarrow{\sim} \mathbb{C}U$, it is also true for all $f : X \to \mathbb{C}U$.

For each $\mathbb{C}[G]$-endomorphism θ of $\mathbb{C}X$, write $\delta(\chi, \theta)$ for the determinant of the endomorphism θ_V of $\mathrm{Hom}_G(V^*, \mathbb{C})$ induced by θ. In fact, δ is clearly independent of the choice of V associated to χ. One has:

$$R(\chi, f) = \delta(\chi, \lambda \circ f)$$

The determinants δ obeys the following results:

1. $\delta(\chi + \chi', \theta) = \delta(\chi, \theta) + \delta(\chi', \theta)$
2. $\delta(\mathrm{Ind}\chi, \theta) = \delta(\chi, \theta)$
3. $\delta(\mathrm{Infl}\chi, \theta) = \delta(\chi, \theta|_{\mathbb{C}X}H)$
4. $\delta(\chi, \theta\theta') = \delta(\chi, \theta)\delta(\chi, \theta')$
5. $\delta(\chi, f)^\alpha = \delta(\chi^\alpha, \theta^\alpha)$ for all $\alpha \in \mathrm{Aut}(\mathbb{C})$

Here, (1) is trivial, (2) follows from the fact that for all representation W of the subgroup H of G and for all $\mathbb{C}[G]$-module Z, there is a natural isomorphism $\mathrm{Hom}_G(\mathrm{Ind}_H^G W) \simeq \mathrm{Hom}_H(W, Z)$, where, in the term on the right, Z is considered a H-module. (3) refers to the following situation:

Suppose $k \subseteq K' \subseteq K$ with K'/k Galois. Denote by H, the group $\mathrm{Gal}(K/K')$ and X' the abelian group relative to K'. We then embed X' in X by $w' = \sum_{w'|w}[w : w']w = \sum_{h\in H} w_0^h$ where $[w : w']$ is the degree of the local extension $K_w/K'_{w'}$, and w_0 is an arbitrary place of K lying above w. It is this normalisation that makes the following diagram commutative :

$$\begin{array}{ccc} U & \xrightarrow{\lambda} & \mathbb{R}X \\ \uparrow & & \uparrow \\ U' & \xrightarrow{\lambda'} & \mathbb{R}X' \end{array}$$

where the maps λ, λ' is as defined in §1.3. We then find that $X' = N_H X$ where $N_H = \sum_{h\in H} h \in \mathbb{Z}[G]$, but not, in general $X' = X^H$. Nevertheless, $N_H X$ has finite index in X^H, and thus we have $EX' = EX^H$ for a field E of characteristic 0.

That being said, (3) is evident, the formula (4) is trivial, as for (5), let $\alpha : \mathbb{C} \to \mathbb{C}$ be be an embedding and write $\theta^\alpha = 1 \otimes_\alpha \theta \colon \mathbb{C} \otimes_\alpha \mathbb{C}X \to \mathbb{C} \otimes_\alpha \mathbb{C}X$. X^α is viewed as $\mathbb{C} \otimes_\alpha V$ by the usual identification

$$\mathrm{Hom}_{\mathbb{C}\otimes_\alpha\mathbb{C}[G]}(\mathbb{C}\otimes_\alpha V^*, \mathbb{C}\otimes_\alpha \mathbb{C}X) = \mathbb{C}\otimes_\alpha \mathrm{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X) \tag{1.12}$$

The endomorphism $(\theta^\alpha)_V$ becomes $1 \otimes_\alpha \theta_V$, and the determinant is $(\det \theta_V)^\alpha$.

Statement 2.6.2 now follows from (5) and the obvious relation:

$$A(\chi, f) = A(\chi, f_0)\delta(\chi, \theta)$$

where $\theta = f_0^{-1} f$.

Example 16. *Following the discussion earlier in this section, the conjectures in section* 5 *are still equivalent to the statement in Conj.* 14 *applied to the case* $E = \mathbb{C}$ *with the isomorphism* $f = \lambda^{-1}$. *This gives* $R(\chi, \lambda^{-1}) = \delta(\chi, 1) = 1$, *and one obtains this intrinsic but essentially transcendent formulation of the conjecture due to Stark:*

For each $\alpha \in \mathrm{Aut}(\mathbb{C})$, we conjecture that

$$\frac{c(\chi^\alpha)}{c(\chi)^\alpha} = \delta(\chi^\alpha, \lambda \circ \lambda^{-\alpha})$$

1.5. Reduction to the abelian case and independence of S

We draw immediately from the previous section the following formulae concerning the numbers $A(\chi, f)$ introduced in §1.4 (or, more generally, in Conj. 14, suppose $E \subseteq \mathbb{C}$):

1. $A(\chi + \chi', f) = A(\chi, f) \cdot A(\chi', f)$
2. $A(\mathrm{Ind}\chi, f) = A(\chi, f)$
3. $A(\mathrm{Infl}\chi, f) = A(\chi, f|_{\mathbb{C}X} H)$

This formalism allows one to reduce Stark's conjecture, on the one hand to the case $k = \mathbb{Q}$ (by passing to the Galois closure of K and induction), on the other hand to the case when characters are of dimension 1 (due to the theorem of Brauer, refer to Appendix F.)

Proposition 17. *1. If the conjecture is true for all finite Galois extensions* $K/\mathbb{Q}$, *then it is also true in general.*

2. *If the conjecture is true for all irreducible characters of dimension* 1 *of all Galois extensions* K/k*, then it is also true in general.*

This being said, let us pass to the independence of the conjectures on the choice of S:
The set S fixed in section appears in the conjectures of section through an intermediary such as the L-function as well as the definition of the regulator. In fact, one has the

Proposition 18. *The truth of the conjecture in section §1.4 is independent of the choice of the set* S*.*

Proof. We work with the version in . Suppose S is the initial set and let $S' = S \cup \{\mathfrak{p}\}$, where $\mathfrak{p}$ is a place of k not appearing in S. Denote by U', X', f',etc. the data in section with S replaced with S', as well as $c'(\chi), r'(\chi)$ the initial coefficient and the multiplicity of $L_{S'}(s, \chi)$ at $s = 0$ respectively. Finally, let $A'(\chi, f')$ be the resultant number as seen in section. We also assume that $f'\_\mathbb{C}X = f$. Let

$$B(\chi) = \frac{A'(\chi, f')}{A(\chi, f)}$$

We have to show that

Claim: $B(\chi)^\alpha = B(\chi^\alpha)$ for all $\alpha \in \mathrm{Aut}(\mathbb{C})$.

As in , and the formulae in , we note that it is sufficient to solve for $\chi(1) = 1$. This leads us to distinguish the two cases below. Let $\mathfrak{P}$ be a place of K lying above $\mathfrak{p}$ and $G_\mathfrak{P} \subseteq G$ its decomposition group.

Case-1: χ is not trivial on $G_\mathfrak{P}$

Then we have $(\because \dim_\mathbb{C} V = \dim_\mathbb{C} V^*) r(\chi) = r'(\chi); \mathrm{Hom}_G(V^*, \mathbb{C}X) = \mathrm{Hom}_G(V^*, \mathbb{C}X')$ and $R(\chi, f) = R'(\chi, f')$.

On the other hand, if χ is also not trivial on the inertia group $I_\mathfrak{P}$ of $\mathfrak{P}$, we find that $L_S(s, \chi) = L_{S'}(s, \chi)$, and so $B(\chi) = 1 = B(\chi^\alpha)$, which implies 2.7.4. Suppose to the contrary, $\chi(I_\mathfrak{P}) = 1$, then $c'(\chi) = (1 - \chi(\sigma_\mathfrak{P}))c(\chi)$ and, as a consequence $B(\chi) = (1 - \chi(\sigma_\mathfrak{P}))^{-1}$, so that the claim is trivially true.

Case-2: $\chi(G_{\mathfrak{P}}) = 1$.

Due to (3) property mentioned at the start of this section, it is enough to assume $G_{\mathfrak{P}} = 1$, which is to say that $\mathfrak{p}$ splits completely in the extension K/k. In this case, $L_{S'}(s,\chi) = (1 - N\mathfrak{p}^{-s})^{-1} L_S(s,\chi)$, so $c'(\chi) = c(\chi)\log N\mathfrak{p}$. On the other hand, $r'(\chi) = r(\chi) + 1$, and more precisely, if $\mathfrak{P}^h = \pi\mathcal{O}_K$, for $\pi \in K$:

$$\begin{cases} \mathbb{Q}U' & \simeq \mathbb{Q}U \oplus \mathbb{Q}[G]\cdot\pi \\ \mathbb{Q}X' & \simeq \mathbb{Q}X \oplus \mathbb{Q}[G]\cdot(\mathfrak{P} - \frac{1}{g}N_G w_0) \end{cases}$$

where w_0 is an arbitrary Archimedean place of K, $g = \#S$ and $N_G = \sum_{\sigma\in G}\sigma \in \mathbb{Q}[G]$.

In suitable bases, we obtain matrices for λ' and f' :

$$M(\lambda') = \begin{pmatrix} M(\lambda) & * \\ 0 & \log|\pi|_{\mathfrak{P}}\cdot 1_G \end{pmatrix}; M(f') = \begin{pmatrix} M(f) & * \\ 0 & 1_G \end{pmatrix}$$

As V is of dimension 1, it is easily deduced that the matrix corresponding to the endomorphism $(\lambda' \circ f')_V$ of $\mathrm{Hom}_G(V^*, \mathbb{C}X')$ can be put in the form :

$$M(\lambda') = \begin{pmatrix} M((\lambda\circ f)_V) & * \\ 0 & \log|\pi|_{\mathfrak{P}} \end{pmatrix}$$

where $\det M((\lambda\circ f)_V) = R(\chi, f)$.
Finally, one finds that $B(\chi) = \log|\pi|_{\mathfrak{P}}/\log N\mathfrak{p}$, a rational number which does not depend on χ. This concludes the proof of the proposition. □

1.6. Statement of Gross-Stark Conjecture

This section follows [Gro81][Ven14]

1.6.1. Gross's p-adic regulator

Recall that the definition of Stark regulator crucially depends on the logarithmic map λ defined in previous section. We also aim to find such a map. First, we shall build the the theory of p-adic absolute values.

Definition 19. *For each place* $\mathfrak{P}$ *of* K*, we can define the local absolute value* $|\cdot|_{\mathfrak{P},\mathfrak{p}} : K_{\mathfrak{P}}^{\times} \to \mathbb{Z}_p^{\times}$ *by*

$$|x|_{\mathfrak{P},\mathfrak{p}} = \begin{cases} 1 & \text{if } K_{\mathfrak{P}} \simeq \mathbb{C} \\ \operatorname{sign}(x) & \text{if } K_{\mathfrak{P}} \simeq \mathbb{R} \\ (\mathbb{N}\mathfrak{P})^{-v_{\mathfrak{P}}(x)} & \text{if } \mathfrak{P} \nmid \mathfrak{p} \\ (\mathbb{N}\mathfrak{P})^{-v_{\mathfrak{P}}(x)} \mathbb{N}_{K_{\mathfrak{P}}/\mathbb{Q}_p} & \text{if } \mathfrak{P} \mid \mathfrak{p} \end{cases}$$

Remark 20. *1. It can be shown that the product formula holds for the local absolute values as well. More precisely,*

$$\prod_{\mathfrak{P}} |x|_{\mathfrak{P},p} = 1 \;\forall\; x \in K^{\times}$$

2. The local absolute values are not exactly the same as the usual absolute values. For example, if x *is a totally positive unit, then* $|x|_{\mathfrak{P},p} = 1$ *for all places* $\mathfrak{P}$ *but usually if* $|x|_{\mathfrak{P}} = 1$ *for all places* $\mathfrak{P}$*, then* x *must be a root of unity.*

The second property is a useful property to have. So, we focus our attention to the subgroup

$$(K^{\times})^{-} := \{x \in K^{\times} : |x|_{\mathfrak{P},p} = 1 \;\forall\; \mathfrak{P} \mid \infty\}$$

On this subgroup, we have the property that x *is a root of unity contained in* K *if and only if* $|x|_{\mathfrak{P}} = 1$ *for all finite places* $\mathfrak{P}$ *of* K*. [Gro81, Prop. 1.11]*
The above definition can be interpretated in the following manner as well. If $\tau \in G$ *is the complex conjugation, then*

$$(K^{\times})^{-} = \{x \in K^{\times} : \tau(x) = -x\}$$

Next, fix the finite set S of places of K containing all the infinite places and the places dividing p. Let $U_{S,K}$ be the set of S-units of K and let $U_{S,K}^{-} = U_{S,K} \cap (K^{\times})^{\times}$. Let $Y_{S,K}$ be the free abelian group on the set S and let $X_{S,K}$ be the subgroup of elements of degree 0 as in the previous section. Motivated from the logarithmic map $\lambda : U \to \mathbb{R}Y$, we define our local logarithmic map

$$\begin{aligned} \lambda_p : U_{S,K} &\to \mathbb{Q}_p Y_{S,K} \\ x &\mapsto \sum_{\mathfrak{P} \in S} \log_p |x|_{\mathfrak{P},p} \mathfrak{P} \end{aligned}$$

Due the product formula (Remark 20(1), the image of λ_p lies in $\mathbb{Q}_p X_{S,K}$. We are interested in knowing whether the induced map $\lambda_p : \mathbb{Q}_p U^-_{S,K} \to \mathbb{Q}_p X_{S,K}$ is injective or not. The measure of how far the map is from being injective is quantified through the regulator. First, define

$$\begin{aligned} o_p : U^- &\to X^- \\ x &\mapsto \sum_{\mathfrak{P} \nmid \infty} f_{\mathfrak{P}} v_{\mathfrak{P}}(x) \mathfrak{P} \end{aligned}$$

Tensoring by $\mathbb{Q}_p$ over $\mathbb{Z}$ gives the induced map

$$o_p : \mathbb{Q}_p U^- \to \mathbb{Q}_p X^-$$

The map o_p is an isomorphism (just construct the inverse using the finiteness of the class number of K).

Definition 21. *We can define the Gross p-adic regulator via*

$$R_{p,K,S} = \det(\lambda_p \circ o_p^{-1} | \mathbb{Q}_p X^-)$$

1.6.2. Statement of the Gross-Stark conjecture

Let k be a totally real number field and $\overline{k}$ its algebraic closure. Let E be a field of characteristic 0 and V the finite dimensional vector space over E with an action of G_k. Consider the representation

$$\rho : G_k \to \mathrm{GL}(V)$$

that factors through the Galois group of a finite extension K/k. Such a representation is said to be totally odd if every complex multiplication acts as -1_V.

Fix a prime number p, and fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. This allows us to view χ as taking values in $\mathbb{C}$ or $\mathbb{C}_p$. Let S be a finite set of places of k containing all the infinite places of k. To the representation V, we have the S-depleted L-function

$$L_S(s, \rho) = \prod_{\mathfrak{p} \notin S} \det \left(1 - \sigma_{\mathfrak{p}} \mathbb{N}\mathfrak{p}^{-s} | V^{I_{\mathfrak{p}}}\right)^{-1} \tag{1.13}$$

Let S also contain all the divisors of p. Let

$$\omega : G(k(\mu_{2p})/k) \to (\mathbb{Z}/2p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$$

be the Teichmuller character. If $\alpha : E \to \mathbb{C}_p$ is an embedding, then V^α denotes the complex representation obtained by change of base. We have the following interpolation formula:

$$L_{p,S}(\omega^{1-n} \otimes V^\alpha, n) = a_S(V,n)^\alpha$$

where $a_S(V,n)$ is obtained via the relation

$$L_S(V^\beta, n) = a_S(V,n)^\beta$$

with $\beta : E \to \mathbb{C}$ an embedding.

The p-adic L function $L_p(\omega \otimes V^\alpha, s)$ is non-zero if and only if V is totally odd. Next, the Taylor expansion of $L_S(V^\beta, s)$ and $L_{p,S}(\omega \otimes V^\alpha, s)$ at $s = 0$ gives

- $L_S(V^\beta, s) \sim L(V^\beta)s^{r(V^\beta)}$
- $L_{p,S}(\omega \otimes V^\alpha, s) \sim L_p(V^\alpha)s^{r_p(V^\alpha)}$

Definition 22. *Define the regulators*

- $R(V^\beta) = \det\left(1 \otimes \lambda \circ f^{-1} | (V^\beta \otimes \mathbb{C}X^-)^G\right)$
- $R_p(V^\alpha) = \det\left(1 \otimes \lambda_p \circ o_p^{-1} | (V^\alpha \otimes \mathbb{C}X^-)^G\right)$

It can be shown that there is an algebraic number $A(V) \in E^\times$ such that for all embeddings $\beta : E \to \mathbb{C}$ both $r(V^\beta) = r(V)$ and $L(V^\beta) = R(V^\beta)A(V)^\beta$.

Conjecture 23. *For all embeddings $\alpha : E \to \mathbb{C}_p$ we have*

1. $r_p(V^\alpha) = r(V)$
2. $L_p(V^\alpha) = R_p(V^\alpha)A(V)^\alpha$

This conjecture can be reformulated as

Conjecture 24. *1.* $\mathrm{ord}_{s=0} L_{S,p}(\omega\chi, s) = r(\chi)$

2. $$\lim_{s\to 0} L_{S,p}(\omega\chi, s)/s^{r(\chi)} = \left((-1)^{r(\chi)} \prod_{\substack{\mathfrak{p}\in S \\ \chi(\mathfrak{p})=1}} f_\mathfrak{p}\right) \frac{h_K}{h_k} \frac{1}{|\mu(K)|Q} \prod_{\substack{\mathfrak{p}\in S \\ \chi(\mathfrak{p})\neq 1}} (1 - \chi(\mathfrak{p}))$$

2. Cohomological interpretation

The notation from this chapter onwards follows [DDP11]. So, instead of K/k we deal with H/F defined below. Also, we consider the case $\dim V = 1$.

Let F be a totally real field, and

$$\chi : G_F :\to \overline{\mathbb{Q}}^{\times}$$

be a totally odd character of the absolute Galois group of F. Let H be the cyclic extension of F cut out by $\mathrm{Ker}(\chi)$ (in fact more is true, H is a CM extension as well). χ can be seen as operating on the ideals of F via $\chi(\mathfrak{p}) = 0$ if $\mathfrak{p}$ is ramified in H/F and $\chi(\mathfrak{p}) = \chi(\mathrm{Frob}(\mathfrak{p}, H/F))$ if $\mathfrak{p}$ is unramified in H/F.

Next, fix a prime number p, and embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and view χ as having values in $\mathbb{C}$ or $\mathbb{C}_p$. Let $V = E(\chi)$ with E a finite extension of $\mathbb{Q}_p$ containing all values of χ. Substitute V with χ. We wish to reformulate $R_p(\chi)$ cohomologically.

For sake of convenience and completeness, we will restate the problem statement again.

Consider a finite set of places S of F containing all the infinite places. Then, the S-depleted L-function

$$L_S(s, \chi) = \sum_{\substack{\gcd(\mathfrak{a}, v)=1 \\ \forall\ \mathfrak{p} \in S}} \chi(\mathfrak{a}) \mathbb{N}\mathfrak{a}^{-s} = \prod_{\mathfrak{p} \notin S} \left(1 - \chi(\mathfrak{p}) \mathbb{N}\mathfrak{p}^{-s}\right)^{-1}$$

convergent for $\mathrm{Re}(s) > 1$ and has a holomorphic continuation to all of $s \in \mathbb{C}$ for nontrivial χ. Due to [DR80] we know of the existence of a continuous E-valued function $L_{S,p}(\chi\omega, s)$ with $s \in \mathbb{Z}_p$ characterised by the interpolation property at negative integers $n \leq 0$:

$$L_{S,p}(\chi\omega, n) = L_S(\chi\omega^n, n)$$

A theorem of Siegel shows that $L_S(\chi, n)$ is algebraic and using the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ we can view the values to be p-adic. In fact, the function $L_{S,p}(\chi\omega, s)$

is meromophic on $\mathbb{Z}_p$ and regular as long as $\chi \neq \omega^{-1}$.

Let S_p be the set of places of F above p. We can take S to be $S_p \cup S_\infty \cup \{v : v \text{ ramified }\}$. We can partition S_p into

$$R = \{\mathfrak{p} \mid p : \chi(\mathfrak{p}) = 1\}, \quad R' = \{\mathfrak{p} \mid p : \chi(\mathfrak{p}) \neq 1\}$$

By the useful observation we made in Theorem 10 we can deduce that $r := r_S(\chi) = \#R$. Gross conjectured that

$$\text{ord}_{s=0} L_{S,p}(\chi\omega, s) = r_S(\chi) = r$$

Remark 25. *If $T = S \backslash R$, then*

$$L_S(\chi, s) = \left(\prod_{\mathfrak{p} \in R} 1 - \mathbb{N}\mathfrak{p}^{-s} \right) L_T(\chi, s)$$

Hence, $L_S(\chi, s) = 0$ at $s = 0$ with order $r_S(\chi)$. By the interpolation property, the order of vanishing of $L_{S,p}(\chi\omega, s)$ at $s = 0$ is atleast r.

For the rest of the exposition, we shall assume that $r_S(\chi) = 1$. In words, this means there is an unique $\mathfrak{p} \mid p$ such that $\chi(\mathfrak{p}) = 1$. In our case, we have

$$\text{ord}_{s=0} L_{S,p}(\chi\omega, s) \geq 1$$

Next, let

$$U_\chi := \left(\mathcal{O}_{H,S}^\times \otimes E\right)^{\chi^{-1}} := \{u \in \mathcal{O}_{H,S}^\times \otimes E : \sigma u = \chi^{-1}(\sigma) u \forall\, \sigma \in G\}$$

Galois equivariant form of Dirichlet's S-unit theorem tells us that U_χ is a finite dimensional E-vector space such that

$$\dim_E U_\chi = r_S(\chi) = 1$$

Fix a vector $0 \neq u_\chi \in U_\chi$ and choose a prime $\mathfrak{P}$ of H lying above $\mathfrak{p}$. This allows us to define two E-linear maps

$$o_{\mathfrak{P}} : U_\chi \to E, u \otimes x \mapsto x \text{ord}_{\mathfrak{P}}(u)$$

$$\ell_{\mathfrak{P}} : U_\chi \to E, u \otimes x \mapsto x \log_p(\mathbb{N}_{H_{\mathfrak{P}}} / \mathbb{Q}_p(u))$$

Definition 26. *Following Greenberg, the $\mathscr{L}$-invariant attached to χ is defined via the ratio*

$$\mathscr{L}(\chi) := -\frac{\ell_{\mathfrak{P}}(u_\chi)}{o_{\mathfrak{P}}(u_\chi)}$$

Remark 27.

The $\mathscr{L}$-invariant does not depend on the choice of the vector u_χ. Indeed, if u'_χ is another non-zero vector, then due to the 1-dimensionality of U_χ as a E-vector space we have $u'_\chi = \pi u_\chi$ with $\pi \in E^\times$. Thus, both the numerator and denominator have the extra factor π which cancels out.

The $\mathscr{L}$-invariant is also independent of the choice of the prime $\mathfrak{P}$ above $\mathfrak{p}$. Indeed, if $\mathfrak{P}'$ were another prime, then using transitivity of G on $\mathfrak{P} \mid \mathfrak{p}$ we have $\mathfrak{P}' = \sigma\mathfrak{P}$ for some $\sigma \in G$. Consequently, $o_{\sigma(\mathfrak{P})} = o_{\mathfrak{P}}(\sigma^{-1}u_\chi) = o_{\mathfrak{P}}(\chi(\sigma)u_\chi) = \chi(\sigma)o_{\mathfrak{P}}(u_\chi)$, and $\ell_{\sigma\mathfrak{P}}(u_\chi) = \chi(\sigma)\ell_{\mathfrak{P}}(u_\chi)$ as well. Hence, the ratio is unaffected by the choice.

We are now ready to state Gross's conjecture for our purposes.

Conjecture 28 (Gross)**.** *Let F be a totally real field, H a totally complex extension of F, and $\chi : \mathrm{Gal}(H/K) \to \mathbb{C}^\times$ a character of conductor $\mathfrak{n}$. If $S = R \cup \{\mathfrak{p}\}$ and $r_S(\chi) = 1$, then one can show that*

$$L'_{S,p}(\chi\omega, 0) = \mathscr{L}(\chi) L_R(\chi, 0)$$

To state the main theorem of DDP, we need to introduce some notation.

Definition 29.

$$\mathscr{L}_{an}(\chi, s) := \frac{-L_{S,p}(\chi\omega, 1-s)}{L_R(\chi, 0)}$$

$$\mathscr{L}_{an}(\chi) := \frac{L'_{S,p}(\chi\omega, 0)}{L_R(\chi, 0)} = \mathscr{L}'_{an}(\chi, 1)$$

This definition allows to rephrase the conjecture to asking whether $\mathscr{L}_{an}(\chi) = \mathscr{L}(\chi)$. The main theorem of DDP says that

Theorem 30. *Assuming that Leopoldt's conjecture holds for F, and the assumptions*

1. *If $|S_p| > 1$, then the conjecture is true for all χ.*
2. *If $|S_p| = 1$ and furthermore*

$$\mathrm{ord}_{k=1}(\mathscr{L}_{an}(\chi, k) + \mathscr{L}_{an}(\chi^{-1}, k)) = \mathrm{ord}_{k=1}\mathscr{L}_{an}(\chi^{-1}, k) \tag{2.1}$$

Then, the conjecture holds for both χ and χ^{-1}.

2.1. Cohomological interpretation

Let

$$\varepsilon_{cyc} : G_F \to \mathbb{Z}_p^\times$$

be the cyclotomic character. If E is a finite extension of $\mathbb{Q}_p$ that contains the values of χ as usual, then by $E(\chi)$ we will denote the space E on which G_F acts via the continuous action

$$\sigma \cdot x = \chi(\sigma)x$$

Similarly, $E(1)$ is equipped by the continuous action of the cyclotomic character. Thus, $E(\chi)(1)$ has a continuous action of $\chi\varepsilon_{cyc}$, and $E(\chi^{-1})$ the action via χ^{-1}.

2.2. Local Cohomology groups

Let v be a place of F, $G_v \simeq G_{F_v}, I_v \subseteq G_v$ be choice of decomposition group and inertia group at v.

Let $\mathfrak{p}_E = \langle \pi \rangle$ be the maximal ideal of $\mathcal{O}_E$. Tate's local duality gives a perfect pairing

$$\langle , \rangle_{v,n} : H^1(F_v, \mathcal{O}_E/\pi^n(\chi^{-1})) \times H^1(F_v, \mathcal{O}_E/\pi^n(\chi)(1)) \to \mathcal{O}_E/\pi^n$$

Taking limit $n \to \infty$ and then tensoring with E leads to the perfect pairing

$$\langle , \rangle_v : H^1(F_v, E(\chi^{-1})) \times H^1(F_v, E(\chi)(1)) \to E \tag{2.2}$$

Definition 31. *If M is a G_F-module, then the inflation-restriction sequence gives*

$$0 \longrightarrow H^1(G_v/I_v, M^{I_v}) \longrightarrow H^1(F_v, M) \xrightarrow{\mathrm{res}_{I_v}} H^1(I_v, M)^{G_v/I_v}$$

The unramified classes classes are exactly the classes of $H^1(F_v, M)$ that lie in the kernel of res_{I_v}.

If $\chi(G_v) \neq 1$, then G_v/I_v is a pro-cyclic group. Hence,

$$\begin{aligned} H^1(G_v/I_v, \mathcal{O}_E/\pi^n(\chi^{-1})^{I_v}) &= \widehat{H}^{-1}(G_v/I_v, \mathcal{O}_E/\pi^n(\chi^{-1})^{I_v}) \\ &= (\mathcal{O}_E/\pi^n(\chi^{-1})^{I_v})/(\chi^{-1}(v) - 1) \end{aligned}$$

Thus, the quotient has bounded size independent of n. Or equivalently, if we take limit over n, then the limit has torsion. Consequently, tensoring with E tells us that $H^1(G_v/I_v, E(\chi^{-1})^{I_v}) = 0$ and hence there are no unramified classes.

Assume $\chi(G_v) = 1$. Then,

1. $H^1(F_v, E(\chi^{-1})) = H^1(F_v, E) = \mathrm{Hom}_{cts}(G_v, E)$ contains an unramified class
$$\kappa_{unr} : \mathrm{Gal}(F_v^{unr}/F_v) \to \mathcal{O}_E, \quad \mathrm{Frob}_v \mapsto 1$$
2. If $v \mid p$, then we have a ramified class, namely the restriction of the logarithm of the cyclotomic character to G_v. In particular, we are concerned with
$$\kappa_{cyc} = \log_p(\varepsilon_{cyc}) \in H^1(F, E)$$
3. Kummer theory gives a connecting homomorphism (which is an isomophism)
$$\delta_{v,n} : F_v^\times \otimes \mathbb{Z}/p^n\mathbb{Z} \to H^1(F_v, \mathbb{Z}/p^n\mathbb{Z}(1))$$
If we let $F_v^\times \widehat{\otimes} E := (\varprojlim_n F_v^\times \otimes \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}_p} E$, then the connecting homomorphism of Kummer theory becomes the isomorphism
$$\delta_v : F_v^\times \widehat{\otimes} E \to H^1(F_v, E(1))$$
4. We can calculate the pairing. Let $u \in F_v^\times \widehat{\otimes} E$, note that
   - $\langle \kappa_{unr}, \delta_v(u) \rangle_v = -\kappa_{unr}((u, \overline{F}_v|F_v)) = -\kappa_{unr}((u, F_v^{unr}|F_v)) = -o_v(u)$
   - This uses some calculation as can be found in [AT90][Neu13][NSW08]
$$\begin{aligned}\langle \kappa_{cyc}, \delta_v(u) \rangle_v &= -(\log_p \circ \varepsilon_{cyc})((u, \overline{F}_v|F_v)) \\ &= -(\log_p \circ \varepsilon_{cyc})(\mathbb{N}_{F_v/\mathbb{Q}_p}(u), \overline{F}_v|\mathbb{Q}_p) \\ &= -\log_p(\mathbb{N}_{F_v/\mathbb{Q}_p}(u^{-1})) \\ &= -\ell_v(u)\end{aligned}$$
   The above observation helps us view $\delta_v(F_v^\times \widehat{\otimes} E)$ as the orthogonal complement to κ_{unr} under the local Tate duality.

[DDP11, see Lemma 1.3] also calculate the dimensions of the two spaces $H^1(F_v, E(\chi)(1))$

and $H^1(F_v, E(\chi^{-1}))$. In fact, the dimensions of both the spaces are same, given by

$$\begin{cases} [F_v : \mathbb{Q}_p] & \chi(G_v) \neq 1, v \mid p \\ [F_v : \mathbb{Q}_p] + 1 & \chi(G_v) = 1, v \mid p \\ 1 & \chi(G_v = 1), v \nmid p\infty \\ 0 & \text{otherwise} \end{cases}$$

2.3. Global Cohomology groups

Recall the definition of unramified class

$$H^1_{unr}(F_v, E(\chi^{-1})) \simeq H^1(G_v/I_v, E(\chi^{-1})^{I_v})$$

The orthogonal complement of the space $H^1_{unr}(F_v, E(\chi^{-1}))$ under the local Tate duality is denoted by

$$H^1_{unr}(F_v, E(\chi)(1)) := \{u \in H^1(F_v, E(\chi)(1)) : \langle \kappa, u \rangle_v = 0 \; \forall \; \kappa \in H^1_{unr}(F_v, E(\chi^{-1}))\}$$

Under the observation

$$H^1_{unr}(F_v, E(\chi^{-1})) = \begin{cases} E \cdot \kappa_{unr} & \chi(G_v) = 1 \\ 0 & \text{otherwise} \end{cases}$$

we have

$$H^1_{unr}(F_v, E(\chi)(1)) = \begin{cases} \mathcal{O}_v^\times \widehat{\otimes} E & \chi(G_v) = 1 \\ H^1(F_v, E(\chi)(1)) & \text{otherwise} \end{cases}$$

Definition 32.

By $H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$ we denote the subgroup of $H^1(F, E(\chi^{-1}))$ consisting of classes unramified outside of $\mathfrak{p}$ and arbitrary at $\mathfrak{p}$.

By $H^1_{[\mathfrak{p}]}(F, E(\chi^{-1}))$ we denote the subgroup of $H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$ consisting of classes unramified outside of $\mathfrak{p}$ and trivial at $\mathfrak{p}$.

The corresponding orthogonal complements under the local Tate duality are denoted by $H^1_{[\mathfrak{p}]}(F, E(\chi)(1)) \subseteq H^1_{\mathfrak{p}}(F, E(\chi)(1)) \subseteq H^1(F, E(\chi)(1))$.

The main result of this section is

Proposition 33. *The map*

$$\delta : U_\chi \to H^1_{\mathfrak{p}}(F, E(\chi)(1))$$

induced by Kummer theory is an isomorphism. In particular, as a E-vector space, $H^1_{\mathfrak{p}}(F, E(\chi)(1))$ has dimension 1.

Proof. This result can be generalised for R of size greater than 1. In that case the dimension of the corresponding cohomology class is $|R|$. I will include the proof in the final draft when I state Gross-Stark conjecture in the general setting. □

If W is a subspace of $H^1(F_{\mathfrak{p}}, E(\chi^{-1}))$, define

$$H^1_{W,\mathfrak{p}}(F, E(\chi^{-1})) \subseteq H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$$

to be the subspace consisting of classes whose image under the map $\mathrm{res}_{I_{\mathfrak{p}}}$ lies in W. The dimension of this new subspace is also of interest to us. The following theorem addresses this question.

Proposition 34. *Suppose $\chi(G_{\mathfrak{p}}) = 1$, and $W \subseteq H^1(F_{\mathfrak{p}}, E)$ is a subspace containing the unramified cocycle κ_{unr}. Then,*

$$\dim_E H^1_{W,\mathfrak{p}}(F, E(\chi^{-1})) = \dim_E W - 1$$

In particular,

$$\dim_E H^1_{\mathfrak{p}}(F, E(\chi^{-1})) = [F_{\mathfrak{p}} : \mathbb{Q}_p]$$

Proof. This result can also be easily generalised using the same arguments as used in DDP. I will include it in the final thesis. □

2.4. Formula for $\mathscr{L}$ invariant

Definition 35. *If W_{cyc} is the subspace of $H^1(F_{\mathfrak{p}}, E)$ spanned by the classes κ_{unr} and κ_{cyc}, define*

$$H^1_{\mathfrak{p},cyc}(F, E(\chi^{-1})) := H^1_{\mathfrak{p},W_{cyc}}(F, E(\chi^{-1})) \tag{2.3}$$

By the previous proposition, the space $H^1_{\mathfrak{p},cyc}(F, E(\chi^{-1}))$ is 1-dimensional over E. Thus, any non-trivial element κ in this space is of the form

$$\mathrm{res}_{I_{\mathfrak{p}}}(\kappa) = x\kappa_{unr} + y\kappa_{cyc}$$

for some $x, y \in E$. $y \neq 0$ for it contradicts the dimension when W_{cyc} is spanned by just κ_{unr}. As the space is 1-dimensional, the choice of κ does not change the ratio y/x.

By the reciprocity law of Global Class Field theory, we have

$$\begin{aligned}\langle \kappa, \delta(u_\chi) \rangle &= \sum_v \langle \mathrm{res}_{I_v} \kappa, \delta_v(u_\chi) \rangle_v \\ &= \langle \mathrm{res}_{I_\mathfrak{p}} \kappa, \delta_\mathfrak{p}(u_\chi) \rangle_\mathfrak{p} \\ &= x \langle \kappa_{unr}, \delta_\mathfrak{p}(u_\chi) \rangle_\mathfrak{p} + y \langle \kappa_{cyc}, \delta_\mathfrak{p}(u_\chi) \rangle_\mathfrak{p} \\ &= -x \cdot o_\mathfrak{p}(u_\chi) + y \ell_\mathfrak{p}(u_\chi)\end{aligned}$$

But $\langle \kappa, \delta(u_\chi) \rangle = 0$ by definition. Hence, $\mathscr{L}(\chi) = -x/y$.

Conjecture 36. *The above observation allows us to reduce our theorem to the following:*

There exists a nontrivial class $\kappa \in H^1_{\mathfrak{p},cyc}(F, E(\chi^{-1}))$ *satisfying*

$$\mathrm{res}_{I_\mathfrak{p}}(\kappa) = x\kappa_{unr} + y\kappa_{cyc}$$

3. Λ-adic Hilbert modular forms

Main references are [DDP11][Shi78][Gar90]

3.1. Hilbert Modular Forms

Let F be a totally real number field of degree $n = [F : \mathbb{Q}]$. The embeddings be $\tau_1, \dots, \tau_n$. If $a \in \mathcal{O}_F$, then a can be seen as an element of $F \hookrightarrow \mathbb{R}$ via the tuple $a = (a_i := \tau_i a)_i$.

Let ψ be a narrow ray class character modulo $\mathfrak{b}$ with sign $r \in \mathbb{F}_2^n$. If $\alpha \in \mathcal{O}_F$ is relatively prime to $\mathfrak{b}$, we can define a character associated to ψ by

$$\psi_f : (\mathcal{O}_F/\mathfrak{b})^\times \to \overline{\mathbb{Q}}^\times, \ \ \alpha \mapsto \operatorname{sign}(\alpha)^r \psi(\langle \alpha \rangle)$$

Fix an integer k. Let $\lambda \in \mathrm{Cl}^+(F)$ be an ideal class, choose a representative fractional ideal $\mathfrak{t}_\lambda$. Let $\mathrm{GL}_2^+(F)$ denote the 2×2 matrices with elements from F such that the determinant is totally positive (all galois conjugates are positive). Define the level

$$\Gamma_\lambda := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) : a, d \in \mathcal{O}_F, b \in \mathfrak{t}_\lambda^{-1}\mathfrak{d}^{-1}, c \in \mathfrak{b}\mathfrak{t}_\lambda\mathfrak{d}, ad - bc \in \mathcal{O}_F^\times \right\}$$

The space $M_k(\mathfrak{b}, \psi)$ of Hilbert modular forms of level $\mathfrak{b}$ and character ψ consists of functions $f = (f_\lambda)_{\lambda \in \mathrm{Cl}^+(F)}$ with

$$f_\lambda : \mathcal{H}^n \to \mathbb{C}$$

such that each function f_λ satisfies

$$f_\lambda|_\gamma = \psi_f(a) f_\lambda$$

for all $\gamma \in \Gamma_\lambda$ where the slash operator $|_\gamma$ is defined to be

$$f_\lambda|_\gamma(z) := \det(\gamma)^{k/2}(cz+d)^{-k} f_\lambda(\gamma z)$$

$$(cz+d)^k := \prod_{i=1}^{n} (c_i z_i + d_i)^k$$

$$\det(\gamma)^{k/2} := \prod_{i=1}^{n} \det(\gamma_i)^{k/2}$$

$$\gamma z := \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \cdots, \frac{a_n z_n + b_n}{c_n z_n + d_n} \right)$$

The function $f \in M_k(\mathfrak{b}, \psi)$ must also satisfy

$$S(\mathfrak{m})f = \psi(\mathfrak{m})f \ \forall \ \gcd(\mathfrak{m}, \mathfrak{b}) = 1 \tag{3.1}$$

It can be shown that f_λ has a Fourier expansion

$$f_\lambda(z) = a_\lambda(0) + \sum_{\substack{b \in \mathfrak{t}_\lambda \\ b >> 0}} a_\lambda(b) \exp(2\pi i \mathrm{Tr}_{F/\mathbb{Q}}(x)) \tag{3.2}$$

Definition 37. *The coefficients $a_\lambda(b)$ are called unnormalised Fourier coefficients of f. We define the normalised Fourier coefficients $c(\mathfrak{m}, f), c(0, f)$ to be*

$$c(\mathfrak{m}, f) := a_\lambda(b)\mathbb{N}\mathfrak{t}_\lambda^{-k/2}, \ \ c(0, f) := a_\lambda(0)\mathbb{N}\mathfrak{t}_\lambda^{-k/2}$$

where an integral ideal $\mathfrak{m} = b\mathfrak{t}_\lambda^{-1}$ for a totally positive element b and an unique λ.

Remark 38. *Note that the definition does not depend on the choice of b. Indeed, any other choice of b would differ by a totally positive unit ϵ, and modularity condition would imply $f_\lambda(\epsilon z)\mathbb{N}\epsilon^{k/2} = f_\lambda(z)$.*

Definition 39. *If for each $\gamma \in \mathrm{GL}_2(F)^+$ and $\lambda \in \mathrm{Cl}^+(F)$, the function $f|_\gamma$ has constant term 0, then we say f is a cusp form. The space of cusp forms of weight k, level $\mathfrak{b}$ and character ψ is denoted by $S_k(\mathfrak{b}, \psi)$.*

3.2. Eisenstein series

A standard example of Hilbert modular forms come from Eisenstein series associated to two narrow ray class characters.

Let $\mathfrak{a}, \mathfrak{b}$ be two integral ideals of F, η, ψ be two narrow ray class characters modulo $\mathfrak{a}, \mathfrak{b}$ respectively. Also, suppose the signs of η, ψ are q, r satisfying

$$q + r \equiv (k, k, \dots, k) \mod 2\mathbb{Z}^n$$

Then it can be shown that [DDP11, Proposition 2.1][Shi78, Proposition 3.4] there exists $E_k(\eta, \psi) \in M_k(\mathfrak{ab}, \eta\psi)$ such that

$$c(m, E_k(\eta, \psi)) = \sum_{\mathfrak{r}|\mathfrak{m}} \eta(\mathfrak{m}/\mathfrak{r})\psi(\mathfrak{r})\mathbb{N}\mathfrak{r}^{k-1} \tag{3.3}$$

In fact, the constant term of the Eisenstein series can be computed explicitly[DDP11, Proposition 2.1], as seen in the following

$$c_\lambda(0, E_k(\eta, \psi)) = \begin{cases} 2^{-n}\eta^{-1}(\mathfrak{t}_\lambda)L_S(\psi\eta^{-1}, 1-k) & k > 1, \mathfrak{a} = 1, \\ 0 & k > 1, \mathfrak{a} \neq 1, \\ 2^{-n}\eta^{-1}(\mathfrak{t}_\lambda)L_S(\psi\eta^{-1}, 0) & k = 1, \mathfrak{a} = 1, \mathfrak{b} \neq 1, \\ 2^{-n}\psi^{-1}(\mathfrak{t}_\lambda)L_S(\eta\psi^{-1}, 0) & k = 1, \mathfrak{a} \neq 1, \mathfrak{b} = 1, \\ 2^{-n}(\eta^{-1}(\mathfrak{t}_\lambda)L_S(\psi\eta^{-1}, 0) + \psi^{-1}(\mathfrak{t}_\lambda)L_S(\eta\psi^{-1}, 0)) & k = 1, \mathfrak{a} = 1, \mathfrak{b} = 1, \\ 0 & k = 1, \mathfrak{a}, \mathfrak{b} \neq 1 \end{cases}$$

3.3. Construction of cusp form

Definition 40. *Whenever $L(\psi, 1-k) \neq 0$, the normalised Eisenstein series can be defined as*

$$G_k(1, \psi) := \frac{2^n}{L(\psi, 1-k)} E_k(1, \psi) \tag{3.4}$$

Using the values of $E_k(1, \psi)$ as in the last proposition, we observe that $c_\lambda(G_k(1, \psi), 0) = 1$.

Recall that $\chi : G_F \to \overline{\mathbb{Q}}^\times$ is a character of conductor $\mathfrak{n}$ and $\chi(\mathfrak{p}) = 1$. Let

$$\mathfrak{n}_R = \operatorname{lcm}\left(\mathfrak{n}, \prod_{\mathfrak{q}|p, \mathfrak{q}\neq\mathfrak{p}} \mathfrak{q}\right), \quad \mathfrak{n}_S = \operatorname{lcm}\left(\mathfrak{n}, \mathfrak{p} \prod_{\mathfrak{q}|p, \mathfrak{q}\neq\mathfrak{p}} \mathfrak{q}\right)$$

We will view χ as a character χ_R(resp. $\chi_S = \chi\omega^{1-k}$) with modulus $\mathfrak{n}_R$(resp. $\mathfrak{n}_S$).

We will concern ourselves with the modular form

$$P_k := E_1(1, \chi_R) G_{k-1}(1, \omega^{1-k}) \in M_k(\mathfrak{n}_S, \chi\omega^{1-k}) \tag{3.5}$$

The modular form $G_{k-1}(1, \omega^{1-k})$ makes sense as can be seen from the functional equation of $L_S(\chi, s)$.

[Wil86]Every modular form in $M_k(\mathfrak{n}_S, \chi\omega^{1-k})$ can be written uniquely as a linear combination of a cusp form and the Eisenstein series $E_k(\eta, \psi)$ with the pair (η, ψ) running over the set J of characters with modulus m_η, m_ψ respectively satisfying

$$m_\eta m_\psi = \mathfrak{n}_S, \quad \eta\psi = \chi\omega^{1-k} \tag{3.6}$$

More concretely,

$$P_k = (\text{cusp form}) + \sum_{(\eta,\psi)\in J} a_k(\eta, \psi) E_k(\eta, \psi) \tag{3.7}$$

As we are interested in constructing a cusp form, we would like to remove the contribution from Eisenstein series in the above expression. This is achieved with the help of an appropriate Hecke operator as will be developed later in this section. We are interested in the coefficients $a_k(\chi, \omega^{1-k})$ and $a_k(1, \chi\omega^{1-k})$ for it turns out that their values are ratios of the classical L-functions and p-adic L-functions. We will record this fact in the following

Proposition 41. *[DDP11, Proposition 2.6, 2.7] If $k \in \mathbb{Z}_{\geq 2}$, then*

$$a_k(1, \chi\omega^{1-k}) = \frac{L_R(\chi, 0)}{L_{S,p}(\chi\omega, 1-k)} = -\mathscr{L}_{an}(\chi, k)^{-1}$$

If $k \in \mathbb{Z}_{>2}$ and $\mathfrak{p}$ is the unique prime above p ($|S_p| = 1$), then

$$a_k(\chi, \omega^{1-k}) = \frac{L_R(\chi^{-1}, 0)}{L_{S,p}(\chi^{-1}, \omega, 1-k)} \langle \mathbb{N}\mathfrak{n} \rangle^{k-1} = -\mathscr{L}_{an}(\chi^{-1}, k)^{-1} \langle \mathbb{N}\mathfrak{n} \rangle^{k-1}$$

Proof. It follows simply by comparing coefficients on both sides of the equation

$$P_k = (\text{cusp form}) + \sum_{(\eta,\psi)\in J} a_k(\eta, \psi) E_k(\eta, \psi)$$

and using the linear independence of characters of the narrow ray class group $\mathrm{Cl}^+(F)$.

□

If $\mathfrak{q}$ is a prime ideal, we denote by $T_\mathfrak{q}, U_\mathfrak{q}$ the Hecke operators. They act on the Eisenstein series in the following manner

$$\begin{aligned} T_\mathfrak{q} E_k(\eta,\psi) &= (\eta(\mathfrak{q}) + \psi(\mathfrak{q})(\mathbb{N}\mathfrak{q})^{k-1})E_k(\eta,\psi) && \mathfrak{q} \nmid \mathfrak{n}_S \\ U_\mathfrak{q} E_k(\eta,\psi) &= (\eta(\mathfrak{q}) + \psi(\mathfrak{q})(\mathbb{N}\mathfrak{q})^{k-1})E_k(\eta,\psi) && \mathfrak{q} \mid \mathfrak{n}_S \\ &= \eta(\mathfrak{q})E_k(\eta,\psi) && \mathfrak{q} \nmid m_\eta \\ &= \psi(\mathfrak{q})(\mathbb{N}\mathfrak{q})^{k-1}E_k(\eta,\psi) && \mathfrak{q} \mid m_\eta \end{aligned}$$

Definition 42. *Remember that $E/\mathbb{Q}_p$ is a finite extension containing the values of χ. Consider the $\mathcal{O}_E$-submodule $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_E) \subseteq M_k(\mathfrak{n}_S, \chi\omega^{1-k})$ consisting of modular forms with the normalised Fourier coefficients lying in the ring $\mathcal{O}_E$. The ordinary projector or Hida's idempotent [Hid93] defined as*

$$e := \lim_{r\to\infty} \left(\prod_{\mathfrak{q}|p} U_\mathfrak{q} \right)^{r!} \tag{3.8}$$

is an idempotent in $\mathrm{End}(M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_E))$*. We can extend it to $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; E)$ E-linearly using the fact that*

$$M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_E) \otimes_{\mathcal{O}_E} E = M_k(\mathfrak{n}_S, \chi\omega^{1-k}; E)$$

It is easy to see that $eE_k(\eta,\psi) = E_k(\eta,\psi)$ if $\gcd(p, m_\eta) = 1$ and 0 otherwise. This allows us to formulate

Proposition 43. *[DDP11, Proposition 2.8] If $P_k^0 = eP_k$, then*

$$P^0 = (\text{ an ordinary cusp form}) + \sum_{(\eta,\psi)\in J^0} a_k(\eta,\psi)E_k(\eta,\psi)$$

where (η,ψ) runs through the set J_0 consisting of the pairs (η,ψ) such that

$$m_\eta m_\psi = \mathfrak{n}_S, \quad \eta\psi = \chi\omega^{1-k}, \gcd(p, m_\eta) = 1 \tag{3.9}$$

Lemma 44. *[DDP11, Lemma 2.9]*

1. *For each $(\eta,\psi) \in J^0$ with $\eta \notin \{1,\chi\}$, we have a Hecke operator $T_{(\eta,\psi)}$ such*

that

$$T_{(\eta,\psi)}E_k(\eta,\psi)=0,\quad T_{(\eta,\psi)}E_1(1,\chi_S)=1$$

2. *If the set $R=S-\{\mathfrak{p}\}$ contains a prime above p, then there is a Hecke operator $T_{(\chi,\omega^{1-k})}$ satisfying*

$$T_{(\chi,\omega^{1-k})}E_k(\chi,\omega^{1-k})=0,\quad T_{(\chi,\omega^{1-k})}E_1(1,\chi_S)=1$$

If F has prime above p other than $\mathfrak{p}$, then set

$$u_k:=\frac{1}{1+\mathscr{L}_{an}(\chi,k)},\quad w_k:=\frac{\mathscr{L}_{an}(\chi,k)}{1+\mathscr{L}_{an}(\chi,k)},\quad v_k:=0$$

If $\mathfrak{p}$ is the unique prime in F above p, then set

$$u_k:=\frac{\mathscr{L}_{an}(\chi,k)^{-1}}{c_k},\quad w_k:=\frac{1}{c_k},\quad v_k:=0\frac{\mathscr{L}_{an}(\chi^{-1},k)^{-1}\langle\mathbb{N}\mathfrak{n}\rangle^{k-1}}{c_k}$$

where

$$c_k=\mathscr{L}_{an}(\chi,k)^{-1}+\mathscr{L}_{an}(\chi^{-1},k)^{-1}\langle\mathbb{N}\mathfrak{n}\rangle^{k-1}+1$$

As a direct corollary to the lemma and the notations above, we have

Theorem 45. *[DDP11, Corollary 2.10] If $H_k:=u_kE_k(1,\chi\omega^{1-k})+v_kE_k(\chi,\omega^{1-k})+w_kP_k^0$, then the modular form*

$$F_k:=\left(\prod_{(\eta,\psi)}T_{(\eta,\psi)}\right)H_k$$

is a cusp form belonging to $S_k(\mathfrak{n}_S,\chi\omega^{1-k})$. The product is over J^0 with $\eta\neq 1$ if F has primes other than $\mathfrak{p}$ above p and the product is over J_0 with $\eta\neq 1,\chi$ if $\mathfrak{p}$ is the only prime in F above p.

3.4. Λ-adic Eisenstein series

Recall that the Iwasawa algebra $\Lambda\simeq\mathcal{O}_E[[T]]$ is topologically generated over $\mathcal{O}_E$ by the functions of the form $k\mapsto u^k$ with $u\in 1+2p\mathbb{Z}_p$. For each $k\in\mathbb{Z}_p$ we have a homomorphism

$$\nu_k:\Lambda\to\mathcal{O}_E,T\mapsto u^{k-1}-1$$

called the specialisation to weight k. $\Lambda_{(k)}$ will denote the localisation of Λ at $\mathrm{Ker}\nu_1$, and sometimes we will view ν_k as a homomorphism from $\Lambda_{(1)} \to E$.

Definition 46. *A family $\mathscr{F} = \{c(\mathfrak{m}, \mathscr{F}), c_\lambda(0, \mathscr{F}), \mathfrak{m}$ integral ideals of $F, \lambda \in \mathrm{Cl}^+(F)\}$ is a Λ-adic form of level $\mathfrak{n}$ and character χ if*
for all finitely many $k \geq 2$, $\exists\ f_k \in M_k(\mathfrak{n}_S, \chi\omega^{1-k}; E)$ such that $\nu_k(c(\mathfrak{m}, \mathscr{F})) = c(\mathfrak{m}, f_k), nu_k(c_\lambda(, \mathscr{F})) = c_\lambda(0, f_k)$ is called a Λ-adic modular form.
Futher, if $\nu_k(\mathscr{F})$ is in $S_k(\mathfrak{n}_S, \chi\omega^{1-k})$ for all but finitely many $k \geq 2$, then we say $\mathscr{F}$ is a Λ-adic cusp form.
The space of Λ-adic modular forms (resp. cusp forms) of level $\mathfrak{n}$ and character χ is denoted by $\mathcal{M}(\mathfrak{n}, \chi)$ (resp. $\mathcal{S}(\mathfrak{n}, \chi)$). By extension of scalars, the elements of $\mathcal{M}(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$ (resp. $\mathcal{S}(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$) are also called Λ-adic modular forms (resp. cusp forms).

The usual Hecke operators $T_\mathfrak{q}, U_\mathfrak{q}$ commute with specialisation. Thus, the action of these operators on the spaces $M_k(\mathfrak{n}_S, \chi\omega^{1-k}), S_k(\mathfrak{n}_S, \chi\omega^{1-k})$ give rise to action in the space $\mathcal{M}(\mathfrak{n}, \chi)$ that preserves $\mathcal{S}(\mathfrak{n}, \chi)$. We also define the ordinary subspaces

$$\mathcal{M}^0(\mathfrak{n}, \chi) := e\mathcal{M}(\mathfrak{n}, \chi), \quad \mathcal{S}^0(\mathfrak{n}, \chi) := e\mathcal{S}(\mathfrak{n}, \chi)$$

It is well known that the ordinary subspaces are finitely generated torsion-free Λ-modules. Let

$$\widetilde{\mathbf{T}} \subseteq \mathrm{End}(\mathcal{M}^0(\mathfrak{n}, \chi)), \quad \mathbf{T} \subseteq \mathrm{End}(\mathcal{S}^0(\mathfrak{n}, \chi))$$

be the Λ algebras generated by the Hecke operators $T_\mathfrak{q}, U_\mathfrak{q}$.
By extension of scalars, the elements of $\mathcal{M}^0(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$ (resp. $\mathcal{S}^0(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$) are also called Λ-adic modular forms (resp. cusp forms).

Proposition 47. *[DDP11, Proposition 3.2] If η, ψ is a pair of narrow ray class characters modulo m_η, m_ψ respectively, such that $\eta\psi$ is totally odd. Then, there exists a Λ-adic modular form*

$$\mathscr{E} \in \mathcal{M}(m_\eta m_\psi, \eta\psi) \otimes_\Lambda \mathcal{F}_\Lambda$$

such that

$$\nu_k(\mathscr{E}(\eta, \psi)) = E_k(\eta, \psi\omega^{1-k})$$

Proof.

$$c(\mathfrak{m}, E_k(\eta, \psi\omega^{1-k})) = \sum_{\mathfrak{r}|\mathfrak{m}} \eta(\mathfrak{m}/\mathfrak{r})\psi(\mathfrak{r})\omega^{1-k}(\mathfrak{r})\mathbb{N}\mathfrak{r}^{k-1}$$
$$= \sum_{\substack{\mathfrak{r}|\mathfrak{m} \\ \gcd(\mathfrak{r},p)=1}} \eta(\mathfrak{m}/\mathfrak{r})\psi(\mathfrak{r})\langle\mathbb{N}\mathfrak{r}\rangle^{k-1}$$

Moreover, if we choose $s \in \mathbb{Z}_p$ such that $\langle\mathbb{N}\mathfrak{r}\rangle = u^s$. Then,

$$\nu_k((1+T)^s) = \langle\mathbb{N}\mathfrak{r}\rangle^{k-1}$$

Thus, the terms on the right hand side can be seen as specialisation of elements in Λ. Moreover, the $L_{S,p}(\eta^{-1}\psi\omega, 1-k)$ can also be seen as specialisation of an element of Λ. Hence, $E_k(\eta, \psi\omega^{1-k})$ can be seen as a specialisation of an appropriate Λ-adic form. This completes the proof. □

Definition 48 (Shifted weight forms)**.** *A family $\mathscr{M}' = \{c(\mathfrak{m}, \mathscr{F}), c_\lambda(0, \mathscr{F}), \mathfrak{m}$ integral ideals of $F, \lambda \in \mathrm{Cl}^+(F)\}$ is a Λ-adic form of level $\mathfrak{n}$ and character χ if for all finitely many $k \geq 2$, $\exists\ f_k \in M_{k-1}(\mathfrak{n}_S, \chi\omega^{1-k}; E)$ such that $\nu_k(c(\mathfrak{m}, \mathscr{F})) = c(\mathfrak{m}, f_k), \nu_k(c_\lambda(, \mathscr{F})) = c_\lambda(0, f_k)$ is called a Λ-adic modular form.*

Proposition 49. *There exists an element $\mathscr{G} \in \mathcal{M}' \otimes_\Lambda \mathcal{F}_\Lambda$ such that*

$$\nu_k(\mathscr{G}) = G_{k-1}(1, \omega^{1-k})$$

Furthermore, if Leopoldt's conjecture holds for F, then the form $\mathscr{G} \in \mathcal{M}' \otimes_\Lambda \Lambda_{(1)}$, and

$$\nu_1(\mathscr{G}) = 1$$

Proof. The existence of $\mathscr{G}$ follows by defining it via

$$\nu_k(c(\mathfrak{m}, \mathscr{G})) = 2^n\zeta_p(F, 2-k)^{-1} \sum_{\substack{\mathfrak{r}|\mathfrak{m} \\ \gcd(\mathfrak{r},p)=1}} \eta(\mathfrak{m}/\mathfrak{r})\psi(\mathfrak{r})\langle\mathbb{N}\mathfrak{r}\rangle^{k-1}, \quad \nu_k(c_\lambda(0, \mathscr{G})) = 1$$

If Leopoldt's conjecture holds, then by a result of Colmez cite Colmez, the p-adic zeta-function $\zeta_p(F, s)$ has a pole at $s = 1$ and thus $\zeta_p(F, 2-k)^{-1}$ is regular at $s = 1$ and vanishes at that point. This completes the proof. □

3.5. Λ-adic cusp form

Very naturally, we want to know if our classical modular forms P_k, P_k^0, H_k and the cusp form F_k can be interpolated p-adically. This is where the slightly ad-hoc condition in the hypothesis comes in handy.

Proposition 50. *[DDP11, Proposition 3.4, Lemma 3.5] Suppose Leopoldt's conjecture holds for F, and*

$$\mathrm{ord}_{k=1}(\mathscr{L}_{an}(\chi, k) + \mathscr{L}_{an}(\chi^{-1}, k)) = \mathrm{ord}_{k=1}\mathscr{L}_{an}(\chi^{-1}, k)$$

Then there exist Λ-adic forms $\mathscr{P} \in \mathcal{M}(\mathfrak{n}, \chi) \otimes \Lambda_{(1)}, \mathscr{P}^0, \mathscr{H} \in \mathcal{M}^0(\mathfrak{n}, \chi) \otimes \Lambda_{(1)}, \mathscr{F} \in \mathcal{S}^0(\mathfrak{n}, \chi) \otimes \Lambda_{(1)}$ such that for all $k \geq 2$

$$\nu_k(\mathscr{P}) = P_k, \nu_k(\mathscr{P}^0) = P^0, \nu_k(\mathscr{H}) = H_k, \nu_k(\mathscr{F}) = F_k$$

In particular, the weight 1 specialisations are

$$\nu_1(\mathscr{P}) = \nu_1(\mathscr{P}^0) = E_1(1, \chi_R), \nu_1(\mathscr{H}) = E_1(1, \chi_S),$$

$$\nu_1(\mathscr{F}) = tE_1(1, \chi_S) \text{ for some } t \in E^\times$$

4. Galois representations

The eigenform $\mathscr{H}$ determines a $\Lambda_{(1)}$ homomorphism

$$\phi_1 : \widetilde{\mathbf{T}} \otimes \Lambda_{(1)} \to E, \quad T \mapsto \nu_1(c(\mathcal{O}_F, T\mathscr{H}))$$

which sends a Hecke operator to its eigenvalue on $\nu_1(\mathscr{H}) = E_1(1, \chi_S)$. The map ϕ_1 can actually be seen as a map from $\mathbf{T} \otimes \Lambda_{(1)} \to E$ as there is a Hecke operator T such that $T\mathscr{H} = \mathscr{F}$ and $\nu_1(\mathscr{F}) = tE_1(1, \chi_S)$. Let $\mathbf{T}_{(1)}$ be the localisation of $\mathbf{T} \otimes \Lambda_{(1)}$ at $\mathrm{Ker}(\phi_1)$. $\mathbf{T}$ is finitely generated as Λ-algebra and $\Lambda_{(1)}$ is Noetherian, so $\mathbf{T} \otimes \Lambda_{(1)}$ is Noetherian and so is $\mathbf{T}_{(1)}$. Moreover, $\mathbf{T}_{(1)}$ is reduced [DK23]. Therefore, the total ring of fractions $\mathcal{F}_{(1)}$ of $\mathbf{T}_{(1)}$ is isomorphic to a product of fields

$$\mathcal{F}_{(1)} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_t$$

where each $\mathcal{F}_i$ is a finite extension of $\mathcal{F}_\Lambda$. Fix a factor $\mathcal{F} := \mathcal{F}_i$. We write $T_\mathfrak{q}$ and $U_\mathfrak{q}$ for the images of the corresponding Hecke operators in $\mathcal{F}_j$ under the projection map $\mathcal{F}_{(1)} \to \mathcal{F}$.

Recall the cyclotomic character

$$\varepsilon_{cyc} : G_F \to \mathbb{Z}_p^\times$$

satisfying

$$\varepsilon_{cyc}(\mathrm{Frob}_\mathfrak{q}) = \mathbb{N}\mathfrak{q} \quad \text{if } \mathfrak{q} \nmid p$$

We can define the Λ-adic cyclotomic character

$$\underline{\varepsilon}_{cyc} : G_F \to \mathbb{Z}_p^\times$$

by

$$\underline{\varepsilon}_{cyc}(\mathrm{Frob}_\mathfrak{q})(k) = \langle \mathbb{N}\mathfrak{q} \rangle^{k-1}$$

4.1. Galois representation attached to ordinary eigenform

A theorem of Wiles says that

Theorem 51. *[DDP11, Lemma 4.1] There exist a continuous irreducible Galois representation*

$$\rho : G_F \to \mathrm{GL}_2(\mathcal{F})$$

such that

1. *The representation ρ is unramified at at all primes $\mathfrak{q}$ outside S, and the characteristic polynomial of $\rho(\mathrm{Frob}_\mathfrak{q})$ is given by*

$$x^2 - T_\mathfrak{q} x + \chi(\mathfrak{q})\langle \mathbb{N}\mathfrak{q}\rangle^{k-1}$$

   In particular, $\det \rho = \chi \varepsilon_{cyc}$

2. *The represenation ρ is odd, i.e. the complex multiplication acts as -1.*

3. *For each $\mathfrak{q} \mid p$, let $G_\mathfrak{q}$ denote the decomposition group at $\mathfrak{q}$. Then,*

$$\rho|_{G_\mathfrak{q}} = \begin{pmatrix} \chi\varepsilon_{cyc}\eta_\mathfrak{q}^{-1} & \star \\ 0 & \eta_\mathfrak{q} \end{pmatrix}$$

   where $\eta_\mathfrak{q}$ is the unramified character of $G_\mathfrak{q}$ satisfying

$$\eta_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}) = U_\mathfrak{q}$$

Let V ne the vector space associated to ρ. For a choice of basis of V, we can represent ρ as

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$$

with $a, b, c, d : G_F \to \mathcal{F}$. Let R denote the image of $\mathbf{T}_{(1)}$ under the projection map $\mathcal{F}_{(1)} \mapsto \mathcal{F}$. Fix a choice of complex conjugation $\delta \in G_F$. Since ρ is totally odd, we can find a basis of V such that

$$\rho(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We fix the basis once and for all. For $x \in R$, let $\overline{x} \in E$ be its reduction modulo the maximal ideal $\mathfrak{m}$ of R.

Theorem 52. *[DDP11, Theorem 4.2] The representation ρ has the following properties:*

1. *For all $\sigma \in G_F$, the elements $a(\sigma), d(\sigma) \in R^\times$, and $\overline{a}(\sigma) = 1, \overline{d}(\sigma) = \chi(\sigma)$. In particular,*
$$\phi_1 \circ a = 1, \quad \phi_1 \circ d = \chi$$
2. *The matrix entry b does not vanish identically on the decomposition group $G_\mathfrak{p}$ at $\mathfrak{p}$.*

4.2. $1 + \epsilon$ specialisation

Let $\nu_{1+\epsilon} : \Lambda_{(1)} \mapsto \widetilde{E} := E[\epsilon]/(\epsilon^2)$ be the map $f \mapsto f(1) + f'(1)\epsilon$.

Recall that $\phi(1)(T_\mathfrak{q}) = \nu_1(c(\mathfrak{q}, T_\mathfrak{q}\mathscr{H})) = T_\mathfrak{q}H_1 = T_q E_1(1, \chi_S) = 1 + \chi_S(\mathfrak{q})$ for $\mathfrak{q} \neq \mathfrak{p}$. The observation is that $H_{1+\epsilon} = \nu_{1+\epsilon}(\mathscr{H})$ can also be written as sum of two characters that lift $1, \chi$.

Definition 53. *Let $\psi_1 : G_F \to \widetilde{E}$ be a character unramified outside p and defined by*
$$\psi_1(\mathfrak{q}) = 1 + v_1\kappa_{cyc}(\mathfrak{q})\epsilon \quad \forall\ \mathfrak{q} \nmid p$$
$$\psi_1(\mathfrak{q}) = 1 \quad \mathfrak{q} \mid p$$
Let $\psi_2 : G_F \to \widetilde{E}$ be a character unramified outside S and defined by
$$\psi_2(\mathfrak{q}) = \chi(\mathfrak{q})(1 + u_1\kappa_{cyc}(\mathfrak{q})\epsilon \quad \forall\ \mathfrak{q} \nmid p$$
$$\psi_2(\mathfrak{q}) = 0 \quad \mathfrak{q} \in S$$

Theorem 54. *[DDP11, Proposition 3.6] The Fourier coefficients of $H_{1+\epsilon}$ satisfy*

1. $c(1, H_{1+\epsilon}) = 1$
2. $c(\mathfrak{q}, H_{1+\epsilon}) = \psi_1(\mathfrak{q}) + \psi_2(\mathfrak{q})$ *if* $\mathfrak{q} \neq \mathfrak{p}$
3. $c(\mathfrak{p}, H_{1+\epsilon}) = 1 + w_1'(\epsilon)$

And, $H_{1+\epsilon}$ is a simultaneuous eigenform for the Hecke operators $T_{\mathfrak{q}}$ for $\mathfrak{q} \notin S$ and $U_{\mathfrak{q}}$ for $\mathfrak{q} \in S$. The eigenvalues are given by the above calculated coefficients.

This lets us define a $\Lambda_{(1)}$ homomorphism

$$\phi_{1+\epsilon} : \widetilde{\mathbf{T}} \otimes \Lambda_{(1)} \to \widetilde{E}, \quad T \mapsto \nu_{1+\epsilon}(c(\mathcal{O}_F, T\mathscr{H}))$$

In fact, $\phi_{1+\epsilon}$ factors through the quotient $\mathbf{T} \otimes \Lambda_{(1)}$ of $\widetilde{\mathbf{T}} \otimes \Lambda_{(1)}$ as there is a Hecke operator T such that $T\mathscr{H} = \mathscr{F}$.

Proof. We shall prove this theorem in the case when $\mathfrak{p}$ is not the only prime in H above p (the other case is done in [DDP11, Proposition 3.6]). Let $\mathfrak{m}$ be an integral ideal of $\mathcal{O}_F$ and write $\mathfrak{m} = \mathfrak{n}\langle p \rangle$ with $\gcd(\mathfrak{n}, \langle p \rangle) = 1$. Note that

$$c(\mathfrak{m}, E_{1+\epsilon}(1, \chi)) = \sum_{\mathfrak{r}|\mathfrak{n}} \chi(\mathfrak{r})(1 + \epsilon\kappa_{cyc}(\mathfrak{r})) \tag{4.1}$$

$$\chi(\mathfrak{r}) = \chi_S(\mathfrak{r}) \qquad \text{if } \mathfrak{p} \nmid \mathfrak{r} \tag{4.2}$$

$$\chi(\mathfrak{r}) = 0 \qquad \text{if } \mathfrak{p} \mid \mathfrak{r} \tag{4.3}$$

Therefore,

$$c(\mathfrak{m}, E_1(1, \chi)) - c(\mathfrak{m}, E_1(1, \chi_S)) = \sum_{\mathfrak{r}|\mathfrak{m}} (\chi(\mathfrak{r}) - \chi_S(\mathfrak{r})) = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) \sum_{\mathfrak{r}|\mathfrak{n}} \chi(\mathfrak{r}) \tag{4.4}$$

Using the same arguments as in [DDP11] we can show that

$$c(\mathfrak{m}, H_{1+\epsilon}) = \left(\sum_{\mathfrak{r}|\mathfrak{n}} \psi_1(\mathfrak{n}/\mathfrak{r})\psi_2(\mathfrak{r}) \right) (1 + w_1'\epsilon)^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}$$

The result follows from this. ☐

As $w(k) = u(k)\mathscr{L}_{an}(\chi, k)$ we have $w_1' = u_1\mathscr{L}_{an}(\chi)$. Thus, we have the following

Theorem 55. *[DDP11, Theorem 3.7] Assuming that Leopoldt's conjecture holds for F, and the assumptions*

1. *If $|S_p| > 1$, then the conjecture is true for all χ.*
2. *If $|S_p| = 1$ and furthermore*

$$\mathrm{ord}_{k=1}(\mathscr{L}_{an}(\chi, k) + \mathscr{L}_{an}(\chi^{-1}, k)) = \mathrm{ord}_{k=1}\mathscr{L}_{an}(\chi^{-1}, k) \tag{4.5}$$

Then there exists a $\Lambda_{(1)}$-homomorphism

$$\phi_{1+\epsilon} : \mathbf{T}_{(1)} \to \widetilde{E}$$

such that

$$\phi_{1+\epsilon}(T_{\mathfrak{q}}) = \psi_1(\mathfrak{q}) + \psi_2(\mathfrak{q}) \quad \mathfrak{q} \notin S \tag{4.6}$$
$$\phi_{1+\epsilon}(U_{\mathfrak{q}}) = \psi_1(\mathfrak{q}) \quad \mathfrak{q} \in R \tag{4.7}$$
$$\phi_{1+\epsilon}(U_{\mathfrak{p}}) = 1 + u_1 \mathscr{L}_{an}(\chi)\epsilon \tag{4.8}$$

4.3. Construction of cocycle

Recall that for each quotient $\mathcal{F}_i$ of $\mathcal{F}_{(1)}$, we have constructed a Galois representation

$$\rho_i : G_F \to \mathrm{GL}_2(\mathcal{F}_i)$$

The product of these representations give us a Galois representation

$$\rho_{(1)} : G_F \to \mathrm{GL}_2(\mathcal{F}_{(1)})$$

The main properties of the Galois representation $\rho := \rho_{(1)}$ is recorded in the following theorem:

Theorem 56. *[DDP11, Lemma 4.3, Theorem 4.4]*

1. *For the chosen complex conjugation δ, we have*

$$\rho(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2. *For all $\sigma \in G_F$, the entries $a(\sigma)$ and $d(\sigma)$ belong to $\mathbf{T}_{(1)}^{\times}$, and*

$$\phi_{1+\epsilon} \circ a = \psi_1 \tag{4.9}$$
$$\phi_{1+\epsilon} \circ d = \psi_2 \tag{4.10}$$

3. *Let $\psi_1 = 1 + \psi'\epsilon$ and define $a'_{\mathfrak{p}} \in E$ by*

$$\phi_{1+\epsilon}(U_{\mathfrak{p}}) = 1 + a'_{\mathfrak{p}}\epsilon$$

Then, there exists a cohomology class $\kappa \in H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$ *such that*

$$\mathrm{res}_{I_\mathfrak{p}} = -a'_\mathfrak{p}\kappa_{unr} + \kappa_{cyc} - \mathrm{res}_{I_\mathfrak{p}}(\psi')$$

Using this theorem, we can complete the proof of the conjecture. Using the last two theorems, we have $\psi'_1 = v_1\kappa_{cyc}$ and $a'_\mathfrak{p} = u_1\mathscr{L}_{an}(\chi)$, and hence obtain a class $\kappa \in H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$ such that

$$\begin{aligned}\mathrm{res}_{I_\mathfrak{p}} &= -a'_\mathfrak{p}\kappa_{unr} + \kappa_{cyc} - \mathrm{res}_{I_\mathfrak{p}}(\psi') \\ &= -u_1\mathscr{L}_{an}(\chi)\kappa_{unr} + \kappa_{cyc}(1 - v_1) \\ &= u_1(-\mathscr{L}_{an}(\chi)\kappa_{unr} + \kappa_{cyc})\end{aligned}$$

Since $u_1 \neq 0$ due the hypothesis, we can replace κ by κ/u_1. This gives a cohomology class $\kappa \in H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$ such that

$$\mathrm{res}_{I_\mathfrak{p}}(\kappa) = -\mathscr{L}_{an}(\chi)\kappa_{unr} + \kappa_{cyc}$$

If we replace by χ^{-1}, then the roles of b and c are reversed and this completes the proof.

5. Work of Dasgupta-Kakde-Ventullo

6. Stark's conjectures

A. Dedekind Zeta Function

Let k be a number field, S a finite set of places of k containing the infinite places S_∞ of k. Then, define the Dedekind zeta function for $\mathrm{Re}(s) > 1$ by

$$\zeta_k(s) = \zeta_{k,S_\infty}(s) = \prod_{\mathfrak{p} \notin S_\infty} \left(1 - \mathbb{N}\mathfrak{p}^{-s}\right)^{-1} = \sum_{0 \neq \mathfrak{a} \trianglelefteq \mathcal{O}_k} \frac{1}{\mathbb{N}\mathfrak{a}^s} \tag{A.1}$$

and more generally

$$\zeta_{k,S}(s) = \prod_{\mathfrak{p} \notin S} \left(1 - \mathbb{N}\mathfrak{p}^{-s}\right)^{-1} = \sum_{\substack{0 \neq \mathfrak{a} \trianglelefteq \mathcal{O}_k \\ \gcd(\mathfrak{a},\mathfrak{p})=1 \ \forall \ \mathfrak{p} \in S}} \frac{1}{\mathbb{N}\mathfrak{a}^s} \tag{A.2}$$

The function above can be meromorphically continued to all of $s \in \mathbb{C}$. The functional equation is discussed in Appendix F.

B. Abelian L-functions

References for this section is [Tat84, §1], [Mar77].

Let k be a number field, S a finite set of places of k containing the infinite places S_∞ of k. Let χ be a complex valued function on the ideals of the ring of integers of k. Define the L-function formally by

$$L(s,\chi) = \prod_{\mathfrak{p}\notin S_\infty} \left(1 - \chi(\mathfrak{p})\mathbb{N}\mathfrak{p}^{-s}\right)^{-1} = \sum_{0\neq\mathfrak{a}\trianglelefteq\mathcal{O}_k} \frac{\chi(\mathfrak{a})}{\mathbb{N}\mathfrak{a}^s} \tag{B.1}$$

If χ satisfies the asymptotic condition $\chi(\mathfrak{a}) = O(\mathbb{N}\mathfrak{a}^\sigma)$ for $\sigma \in \mathbb{R}$, then $L(s,\chi)$ converges for $\mathrm{Re}(s) > 1+\sigma$.

For example, when $k = \mathbb{Q}$, we have the Dirichlet characters $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times$ for $f \in \mathbb{Z}_{\geq 2}$. The character can be extended to all of $\mathbb{Z}$ by letting $\chi(a) = 0$ if $\gcd(a,f) \neq 1$. For a general k, fix an integral ideal $\mathfrak{f}$ of k and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_\mathfrak{f}^\times \longrightarrow k_\mathfrak{f}^\times \longrightarrow I_\mathfrak{f} \longrightarrow C_\mathfrak{f} \longrightarrow 0$$

where

$$\mathcal{O}_\mathfrak{f}^\times = \{x \in \mathcal{O}_k^\times : x \equiv 1 \mod \mathfrak{f}\}$$

$$k_\mathfrak{f}^\times = \{x \in k^\times : x \equiv 1 \mod \mathfrak{f}\}$$

$$I_\mathfrak{f} = \{\mathfrak{a} \in I : \mathfrak{a} \equiv 1 \mod \mathfrak{f}\}$$

and $C_\mathfrak{f}$ the quotient of $I_\mathfrak{f}$ by the principal ideals generated by elements of $k_\mathfrak{f}^\times$. We want to get a character of $I_\mathfrak{f}$ through a character of $C_\mathfrak{f}$

For $k = \mathbb{Q}$, we have $C_f = (\mathbb{Z}/f\mathbb{Z})^\times/\{\pm 1\}$ which does not really correspond to the Dirichlet characters we started with. We thus have to take into consideration the question of sign: if T is a set of real places of k, we denote by $k_{\mathfrak{f},T}^\times$ (resp. $\mathcal{O}_{\mathfrak{f},T}^\times$) the elements of $k_\mathfrak{f}^\times$(resp. $\mathcal{O}_\mathfrak{f}^\times$) that are positive for all places of T. Let $C_{\mathfrak{f},T}$ denote the quotient of $I_\mathfrak{f}$ by the image of $k_{\mathfrak{f},T}^\times$. This is a finite group. To summarise, we have the following commutative diagram whose rows and columns are exact:

$$
\begin{array}{ccccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_T^\times & \longrightarrow & k_T^\times & \longrightarrow & I & \longrightarrow & C_T & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathfrak{f},T}^\times & \longrightarrow & k_{\mathfrak{f},T}^\times & \longrightarrow & I_{\mathfrak{f}} & \longrightarrow & C_{\mathfrak{f},T} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & (\mathcal{O}/\mathfrak{f})^\times/\mathrm{Im}(\mathcal{O}_T^\times) & & \\
 & & & & & & & & \uparrow & & \\
 & & & & & & & & 0 & &
\end{array}
$$

A homomorphism $\chi : C_{\mathfrak{f},T} \to \mathbb{C}^\times$ is seen as a function on I by letting $\chi(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not coprime to $\mathfrak{f}$. We thus have

$$L(s,\chi) := \prod_{\mathfrak{p} \nmid \mathfrak{f}} \left(1 - \chi(\mathfrak{p})\mathbb{N}\mathfrak{p}^{-s}\right)^{-1}$$

The above product converges for $\mathrm{Re}(s) > 1$.

We say that $\chi : \mathcal{C}_{f_T} \to \mathbb{C}^\times$ is *primitive* (where f_T is the conductor of χ), if for all $f'|f$ and $T' \subseteq T$, there exists a χ' such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{C}_{f_T} & \longrightarrow & \mathcal{C}_{f'_{T'}} \\
 & \overset{\chi}{\searrow} & \downarrow \chi' \\
 & & \mathbb{C}^\times
\end{array}
$$

implying $f' = f, T' = T$. By abuse of language, from now on we say $L(s,\chi)$ is primitive if χ is. Consider a function $L(s,\chi)$ non-primitive if it removes a few Euler factors.

We know how to analytically continue $L(s,\chi)$ to the entire complex plane with a functional equation, cf. Appendix F. If $\chi = \mathbf{1}$, $L(s,\chi)$ is equal to ζ_K or one of $\zeta_{k,S}$ depending on whether $f = 1$ or not. If $\chi \neq \mathbf{1}$, we know that $L(s,\chi)$ is holomorphic and $L(1,\chi) \neq 0$.

In terms of ideles The χs constructed correspond to continuous homomorphisms $\mathbb{A}_k^\times \to S^1$ of finite order and trivial on principal ideals of $k^\times$. In effect, an idele $(x_v) \in \mathbb{A}_k^\times$ corresponds to an ideal $\mathfrak{I}_f$ generated by the components $x_\mathfrak{p}$ for $\mathfrak{p} \mid f$.

We are not going to, in these notes, concern ourselves with the more general quasi-characters of $\mathbb{A}_k^\times$.

The theory of ray class fields establishes, for all pair (f, T) as before, the existence of an unique abelian extension $K_{f,T}$ of k- namely ray class field f - such that the following three conditions are satisfied:

1. A prime ideal $\mathfrak{p}$ of k ramifies in $K_{f,T}$ if and only if $\mathfrak{p} \mid f$.

   Notation: If K/k is an abelian extension with a finite Galois group G, $\mathfrak{p}$ a place of k that does not ramify in K/k and $\mathfrak{P}$ a place of K that divides $\mathfrak{p}$, then we note that $\left(\dfrac{\mathfrak{p}}{K/k}\right)$ is an unique element of $G_{\mathfrak{P}} \subseteq G$ (see below) whose reduction modulo $\mathfrak{P}$ is the automorphism $x \mapsto x^{N\mathfrak{p}}$ on the residue field of $\mathfrak{P}$. As G is abelian, the above depends only on $\mathfrak{p}$.

2. The map $\mathfrak{p} \mapsto \left(\dfrac{\mathfrak{p}}{K_{f,T}/k}\right)$ induces an isomorphism-namely the Artin reciprocity :

$$\psi_f : \mathcal{C}_{f,T} \xrightarrow{\sim} \mathrm{Gal}(K_{f,T}/k)$$

3. The norm $N_{K_{f,T}/k}\mathfrak{a}$ of each ideal $\mathfrak{a} \neq 0$ from $K_{f,T}$ prime to f is a principal ideal generated by an element of $k_{f,T}^\times$.

Moreover, for each finite abelian extension K/k, the Galois group G, there exists a pair (f, T) chosen minimally (called the conductor of) K/k such that

1. $K \subseteq K_{f,T}$;
2. The surjection $\psi_{K/k} : \mathcal{C}_{f,T} \xrightarrow{\psi_f} \mathrm{Gal}(K_{f,T}/k) \to G$ is induced from the map $\mathfrak{p} \mapsto \left(\dfrac{\mathfrak{p}}{K/k}\right)$;
3. The kernel $\ker \psi_{K/k}$ forms the class of representatives of the norms of the ideals of K.

By $\widehat{G}$ we denote the characters (of dimension 1) of the group G. Thanks to $\psi_{K/k}$, the elements of $\widehat{G}$ can be interpreted as a character of the type envisaged in earlier section. The conductor of $\chi \in \widehat{G}$ is then that of the fixed field of $\ker \chi \subseteq G$. By writing primitive functions everywhere, we prove the following decomposition ([see CF10, p. 217]; [Wei95, pp. XIII–10]):

$$\zeta_K(s) = \prod_{\chi \in \widehat{G}} L(s, \chi) = \zeta_K(s) \prod_{\chi \neq \mathbf{1}} L(s, \chi) \tag{B.2}$$

C. Linear representations of finite groups

The reference for this section is [Ser77]

Suppose G is a group of finite order g and E a field of characteristic 0. An E-linear representation of G is a homomorphism $\rho : G \to GL(V)$, for a vector space V over E. This amounts to providing V with an $E[G]$-module structure. We can therefore simply talk about the representation V of G.

The character of the representation ρ is a function $\chi = \chi_\rho : G \to E$, such that the trace equals that of action of the automorphism $\rho(x)$ $(x \in G)$ on E. This is a class function (i.e., $\chi(xyx^{-1}) = \chi(y)\ \forall\ x, y \in G$) with $\chi(1) = \dim V$. It takes its values on a cyclotomic extension of $\mathbb{Q}$ contained in E. We denote by $a \mapsto a^*$ the automorphism of the cyclotomic extension of $\mathbb{Q}$ induced by the substitution $\zeta \mapsto \zeta^{-1}$ of roots of unity. For $E \subseteq \mathbb{C}$, we find that $a^* = \bar{a}$ (complex conjugation). Likewise, we write χ^* (or $\bar{\chi}$, if $E \subseteq \mathbb{C}$) for the character obtained by conjugating the values of χ. Two representations of G are isomorphic if and only if they have the same character. This follows from the orthogonality relations between irreducible characters of G (= characters of representations with no proper G-stable subspace), relative to the following scalar product :

$$\langle \chi_1, \chi_2 \rangle_G = \frac{1}{g} \sum_{\sigma \in G} \chi_1(\sigma)\chi_2^*(\sigma) = \frac{1}{g} \sum_{\sigma \in G} \chi_1(\sigma)\chi_2(\sigma^{-1})$$

We note that $\mathbf{1}_G : G \to E$ is just the trivial character corresponding to the dimension of dimension 1. A *virtual character* of G in E is a combination of $\mathbb{Z}$-linear characters of G attached to the representations of G in E.

Properties of $\langle \cdot, \cdot \rangle_G$

Here, the arguments of the scalar product will be that of virtual characters.

1. $\langle \chi_1, \chi_2 \rangle_G \in \mathbb{Z}$

2. $\langle \chi_1 + \chi_2, \chi_3 \rangle_G = \langle \chi_1, \chi_3 \rangle_G + \langle \chi_2, \chi_3 \rangle_G$

$\langle \chi_1, \chi_2 \rangle_G = \langle \chi_2, \chi_1 \rangle_G = \langle \chi_1 \chi_2^*, \mathbf{1}_G \rangle_G$

3. **Frobenius Reciprocity**

   Suppose H is a subgroup of G of order h; ψ a virtual character of H and χ a virtual character of G. So,

$$\langle \psi, \chi|_H \rangle_H = \langle \mathrm{Ind}_H^G \psi, \chi \rangle_G$$

   Here, for $\sigma \in G$, $\mathrm{Ind}_H^G \psi(\sigma) = \dfrac{1}{h} \displaystyle\sum_{\tau \in G, \tau^{-1}\sigma\tau \in H} \psi(\tau^{-1}\sigma\tau)$

   It is the function induced on G by ψ. If ψ is the character of the representation $H \to GL(W)$, then $\mathrm{Ind}\psi$ is that of the induced representation $E[G] \otimes_{E[H]} W$ of G.

Given an $E[G]$-module V and a subgroup H of G, we let $V^H = \{x \in V : \sigma x = x \ \forall \sigma \in H\}$. If W is also an $E[G]$-module, the action of G on $V \otimes_E W$ is given by $\sigma(x \otimes y) = \sigma x \otimes \sigma y$ and on $\mathrm{Hom}_E(V, W)$ by $(\sigma f)(x) = \sigma(f(\sigma^{-1}x))$ so that $\mathrm{Hom}_{E[G]}(V, W) = \mathrm{Hom}_E(V, W)^G$. If V (resp. W) is a representation of G over E of the character χ (resp. ψ), then $\chi\psi$ is a character of $V \otimes_E W$ while the one of $\mathrm{Hom}_E(V, W)$ is $\chi^*\psi$. In fact, we have $V \otimes_E W \simeq \mathrm{Hom}_E(V^*, W)$ where $V^* = \mathrm{Hom}_E(V, E)$ is the dual of V. According to what we have said, the conjugate character χ^* of the character χ is attached to the action of G on V.

From now onwards, the group G given will be the Galois group of the finite extension K/k of global fields. We will always assume the action of G is on the left. However, we sometimes write a^σ instead of σa (for $\sigma \in G, a \in K$). In these cases, the reader should be accustomed to the formula $a^{\sigma\tau} = (a^{tau})^\sigma$ for $\sigma, \tau \in G, a \in K$.

If w is a place of K, G_w is used to denote the decomposition group of w with respect to K/k, i.e., $G_w = \{\sigma \in G : \sigma w = w\}$. If w is non-archimedean, I_w is used to denote the inertia group of w, formed by the elements of G_w that induces trivial automorphism on the residual extension. So, if v is the restriction of w in k, the Galois group of the residual extension of w/v is identified with G_w/I_w and one notes that $\sigma_w \in G_w/I_w$ is the Frobenius automorphism (elevating to a power of Nv on the residue field of w). σ_w generates G_w/I_w.

If w is archimedean, we sometimes write σ_w for the unique generator of G_w. In fact, in the case G_w is of order 2 or 1 depending on whether w is complex extension of a real place or not.

D. Definition and properties of Artin L-functions

Suppose K/k is a finite Galois extension of number fields with Galois group G. Let $\chi : G \to \mathbb{C}$ be a character of a complex representation $G \to GL(V)$. With the notations as in the previous section, for each place $\mathfrak{P}$ of K, the element $\sigma_{\mathfrak{P}} \in G_{\mathfrak{P}}/I_{\mathfrak{P}}$ acts on $V^{I_{\mathfrak{P}}}$. Note that, for $\mathrm{Re}(s) > 1$,

$$L(s,V) = \prod_{\mathfrak{p}} \det(1 - \sigma_{\mathfrak{P}} N\mathfrak{p}^{-s} | V^{I_{\mathfrak{P}}})^{-1} \tag{D.1}$$

where $\mathfrak{p}$ denotes a finite place of k and for each $\mathfrak{p}$, $\mathfrak{P}$ is a place of K dividing $\mathfrak{p}$ (arbitrarily chosen). The $\sigma_{\mathfrak{P}}$ given are conjugates of each other, thus the value of the "characteristic polynomial" of $\sigma_{\mathfrak{P}}$ appearing as a member in the product is independent of the choice of $\mathfrak{P}$.

The same argument shows that $L(s,V)$ remains unchanged if change V by an isomorphic representation. We can therefore write $L(s,\chi)$ without ambiguity instead of $L(s,V)$. In fact, here is an explicit formula due to Artin which depends only on χ :

$$\log L(s,\chi) = \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \frac{\chi(\sigma_{\mathfrak{P}}^n)}{n \cdot N\mathfrak{p}^{ns}} \tag{D.2}$$

where $\chi(\sigma_{\mathfrak{P}}^n) = \frac{1}{|I_{\mathfrak{P}}|} \sum_{\tau \in \sigma_{\mathfrak{P}}^n} \chi(\tau)$

Formal properties:

Once we have shown analytic continuation of $L(s,\chi)$, the following properties become valid for all $s \in \mathbb{C}$.

1. **Additivity**

$$L(s,\chi_1 + \chi_2) = L(s,\chi_1) + L(s,\chi_2)$$

2. **Induction**

$$\begin{array}{c} K \\ {\scriptstyle H}\big| \\ k' \\ \big| \\ k \end{array}$$

For a subgroup H of G and a character χ of H, denote by $\mathrm{Ind}\chi$ the character of G induced from χ. So,

$$L(s, \mathrm{Ind}_H^G \chi) = L(s, \chi)$$

3. **Inflation**

$$\begin{array}{c} K \\ {\scriptstyle H}\big| \\ k' \\ {\scriptstyle G'}\big| \\ k \end{array}$$

For a quotient $G' = G/H$ where H is a distinct subgroup of G and χ a character of G', denote by $\mathrm{Infl}\chi$ the character $G \to G/H \xrightarrow{\chi} \mathbb{C}$. So,

$$L(s, \mathrm{Infl}_{G/H}^G \chi) = L(s, \chi)$$

4. If $\chi(1) = 1$, that is to say that V is of dimension 1, the homomorphism $\chi : G \to \mathbb{C}^\times$ factorises through the abelianisation G^{ab} of G.

E. A theorem of Brauer and Artin's conjecture

A character of G is termed *monomial* if it is induced by a character of degree 1 of a subgroup of G. The theorem of Brauer affirms that all characters of G are integral linear combination of irreducible monomial characters.

Thanks to our discussion in last section, we can deduce that each Artin L-function can be written in the form

$$\prod_i L(s, \psi_i)^{n_i}$$

with $n_i \in \mathbb{Z}$ and ψ_i is a character of degree $\psi_i(1) = 1$ of a suitable subgroup H_i of G. On applying the induction property, we can pass to a quotient H_i of $\ker \psi_i$, so that ψ_i becomes a character of cyclic group.

Let χ be a character of a complex representation of G. One cannot always impose on the integers n_i to be positive. Nevertheless, this decomposition tells us that $L(s, \chi)$ has an analytic continuation to a meromorphic function defined on the entire complex plane.

The conjecture due to Artin says that $L(s, \chi)$ is an entire function, if χ does not contain the trivial character $\mathbf{1}_G$ ([Mar77, pp. I–5]).

F. Functional equation

The main references for this section is [Wei95][God95a][God95b]

Let χ be a character of a complex representation of $G = \mathrm{Gal}(K/k)$.

To begin, complete $L(s,\chi)$ with the gamma factors corresponding to the infinite places of k. Let

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$$

$$\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2\cdot(2\pi)^{-s}\Gamma(s)$$

For each infinite place v of k, choose a place w of K lying above v. If G_w has order 2 (cf. 1.4.4), let χ_- be a non-trivial character. In any case, put $\chi_+ = \mathbb{1}_{G_w}$ and write

$$\chi|_{G_w} = n_+(w)\chi_+ + n_-(w)\chi_-$$

So we have $n_+(w) = \dim V$ and $n_-(w) = \mathrm{codim} V^{G_w}$. Using this decomposition, the local factor L_V does not depend on our choice of w, and is defined by the additivity from the formulas : $\begin{cases} L_V(s,\chi_+) = \Gamma_{\mathbb{C}}(s) & \text{if } V \text{ is complex} \\ L_V(s,\chi_+) = \Gamma_{\mathbb{R}}(s) & \text{if } V \text{ is real} \\ L_V(s,\chi_-) = \Gamma_{\mathbb{R}}(s+1) & \text{if } V \text{ is real} \end{cases}$

If r_2 is the number of complex places of k, we set

$$\begin{aligned} a_1 &= a_1(\chi) = \textstyle\sum_{v \text{ real}} \dim V^{G_w} \\ a_2 &= a_2(\chi) = \textstyle\sum_{v|\infty} \mathrm{codim} V^{G_w} = \sum_{v \text{ real}} \mathrm{codim} V^{G_w} \\ n &= [k:\mathbb{Q}] = \tfrac{1}{\chi(1)}(a_1(\chi) + a_2(\chi) + 2r_2\chi(1)) \end{aligned}$$

More explicitly,

$$\prod_{v|\infty} L_V(s,\chi) = 2^{r_2\chi(1)(1-s)}\mathfrak{p}^{-\frac{a_2}{2}-\frac{s}{2}n\chi(1)}\Gamma(s)^{r_2\chi(1)}\Gamma(s/2)^{a_1}\Gamma\left(\frac{s+1}{2}\right)^{a_2} \tag{F.1}$$

Note that we have $a_i(\chi) = a_i(\overline{\chi})$ for $i = 1, 2, \ldots$

If $\mathfrak{p}$ is a finite place of k, choose a place $\mathfrak{P}$ of K such that $\mathfrak{P}|\mathfrak{p}$. If $I_{\mathfrak{P}} = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ be the sequence of ramification groups of $\mathfrak{P}/\mathfrak{p}$ ([Ser79, ch. IV]). We denote by g_i the cardinality of G_i and put

$$f(\chi, \mathfrak{p}) = \sum_{i=0}^{\infty} \frac{g_i}{g_0} \mathrm{codim} V^{G_i} \tag{F.2}$$

This number does not depend on the choice of $\mathfrak{P}$ and we can show that it is a rational integer ([Ser79, see VI-2]). As we trivially have $f(\chi, \mathfrak{p}) = 0$ if $\mathfrak{p}$ does not ramify in K/k, we can define the Artin conductor of χ by

$$f(\chi) = \prod_{\mathfrak{p}} \mathfrak{p}^{f(\chi,\mathfrak{p})} \tag{F.3}$$

where $\mathfrak{p}$ denotes all the finite places (prime ideals) of k

We put, with the notations as before:

$$\Lambda(s, \chi) = \{|d_k|^{\chi(1)} N f(\chi)\}^{s/2} \prod_{v|\infty} L_V(s, \chi) L(s, \chi) \tag{F.4}$$

where $|d_k| \in \mathbb{Q}$ is the value of the absolute discriminant of k over $\mathbb{Q}$; $Nf(\chi) > 0$ the absolute norm of $f(\chi)$; and for a real positive α and $z \in \mathbb{C}$, we put (here and then) $\alpha^z = \exp(z \log \alpha)$ with $\log \alpha \in \mathbb{R}$.
So, the functional equation of $L(s, \chi)$ can be written as

$$\Lambda(1 - s, \chi) = W(\chi) \Lambda(s, \overline{\chi}) \tag{F.5}$$

with a constant $W(\chi) \in \mathbb{C}^\times$ of modulus 1.

The constant $W(\chi)$-named "Artin's Wurzelzahl" is written as

$$W(\chi) = W_\infty(\chi) \tau(\overline{\chi}) (N f(\chi))^{-1/2} \tag{F.6}$$

where $W_\infty(\chi) = \prod_{v|\infty} i^{\mathrm{codim} V^{G_w}} = i^{-a_2(\chi)}$ and the complex constants $\tau(\overline{\chi})$ are characterised by the following formalism :

1. $\tau(\chi_1 + \chi_2) = \tau(\chi_1)\tau(\chi_2)$
2. $\tau(\mathrm{Ind}_H^G(\chi)) = \tau(\chi) \left((N_{k/\mathbb{Q}} \mathcal{D}(k'/k))^{1/2} i^{m(k'/k)} \right)^{\chi(1)}$

$$\begin{array}{c} K \\ {\scriptstyle H}\Big| \\ k' \\ \Big| \\ k \end{array}$$

where $\mathcal{D}(k'/k)$ is the discriminant ideal of k' on k and $m(k'/k) = \#\{v' : v'|\infty : v' \text{ is a place of } k', G_{v'}(k'/k) \neq \{1\}\}$

3. If χ is of dimension 1, we interpret accordingly as a Dirichlet character of k, so $\tau(\chi)$ is a Gauss sum involved in the functional equation of the abelian L-function (see [MaD], II-2 for the explicit local formulas).

Note that ([Mar77]) we have $W_\infty(\overline{\chi}) = W_\infty(\chi)$ and $f(\overline{\chi}) = f(\chi)$. Finally, we will rewrite the explicit functional equation by using the following identity ([Mar77, p. 49]):

$$W(\chi) = \frac{Nf(\chi)^{1/2}}{\tau(\chi)W_\infty(\chi)} \tag{F.7}$$

The sign of the discriminant $d_k \in \mathbb{Q}$ is $(-1)^{r_2}$. We put

$$\sqrt{d_k} = i^{r_2}|d_k|^{1/2} \in \mathbb{C}$$

With all the notations, here is a explicit version of the functional equation :

$$L(1-s,\chi) = \begin{cases} 2^{r_2\chi(1)} \frac{i^{(a_1+r_2\chi(1))}}{\tau(\chi)\sqrt{d_k}^{\chi(1)}} \pi^{1/2n\chi(1)} \left(\frac{\Gamma(s)}{\Gamma(1-s)}\right)^{r_2\chi(1)} \left(\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\right)^{a_1} \\ \left(\frac{\Gamma((1+s)/2)}{\Gamma((2-s)/2)}\right)^{a_2} B^s L(s,\overline{\chi}) \end{cases} \tag{F.8}$$

where B is a non-zero positive real.

Let us write $c(\overline{\chi})$ (resp. $c_1(\chi)$) for the first non-zero coefficient in the Laurent series expansion of $L(s,\overline{\chi})$ (resp. $L(s,\chi)$) at $s=0$ (resp. $s=1$) and let $r_1(\chi)$ be the multiplicity of $L(s,\chi)$ at $s=1$. Letting $s \to 0$ in equation for $L(1-s,\chi)$, we can finally obtain (recall that $\Gamma(1/2) = \pi^{1/2}$ and that Γ has a simple pole with residue 1 at $s=0$):

$$\frac{c_1(\chi)}{c(\overline{\chi})} = (-1)^{r_1} 2^{r_2\chi(1)+a_1(\chi)} \frac{(\pi i)^{a_2(\chi)+r_2\chi(1)}}{\tau(\chi)\sqrt{d_k}^{\chi(1)}} \tag{F.9}$$

G. Fitting Ideals

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