The Grand ∞ Hotel

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What does one mean by finite and infinite?

Let us see if we can count:

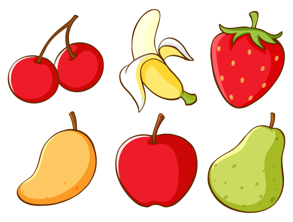


Figure 1. 6 fruits

Figure 2. Population: 1,41,69,22,197

Figure 3. Too many sand particles

Clearly, some things are too numerous to count!

Examples of infinite sets

Have we seen any infinite collections before this ?

- $\mathbb{N} = \{1, 2, \ldots, \}$
- $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$. Like $\frac{1}{2}, \frac{2}{3}$ etc.
- $\mathbb{R}$ like $\sqrt{2}, \pi, e, \pi^e, e^\pi, 2^\pi, etc.$

These sets are clearly infinite. Think for a moment why and read further (take any number in the set you can think of and add 1 to it, you will get another number in the set)

If you can recite π to 7 decimal points you get a 🍬 and 2 🍬🍬 if you can recite e upto 7 decimal points.

14 March is π day!

Are all these infinite sets of the same size?

Let us go through the brilliant thought experiment by Hilbert

Figure 4. David Hilbert

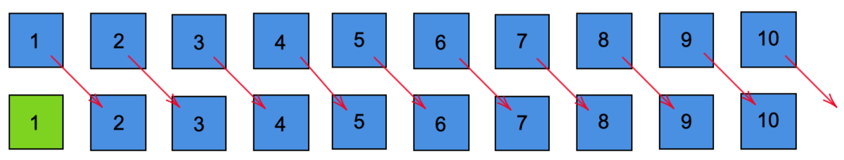
Imagine a Hotel with infinitely many rooms (numbered $1, 2, 3, \ldots$) and each room has exactly one occupant. The hotel is full.

It has a strange ad:

Figure 5. Hilbert's Grand ∞ hotel

Finitely many new guests

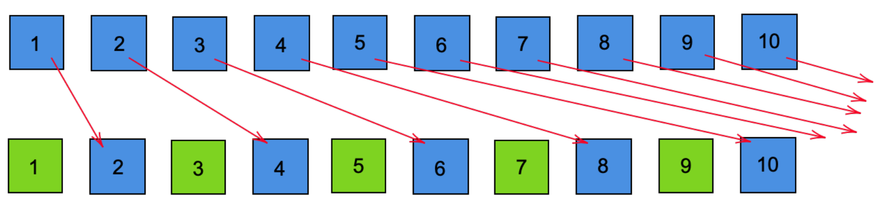
- 1 guest comes and asks for a room. The manager says, "Okay, I have a room for you."



- 100 guests come in and ask for rooms. The manager has no issues accommodating all of them. Why?

A bus with infinite guests

Now, a very long bus with infinitely (countably) many guests come in. They all want a room. The manager takes a moment to think and then arranges rooms for all of them (infinitely many rooms in an already full hotel!).



Impressive!

An infinite number of buses with infinite guests in each of them

Wait a second. The manager now sees infinitely (countably) many buses each with infinitely (countably) many passengers. They all want a room. Now, the manager is a bit stuck, thinks for a minute and fixes the problem.

B1P1	B1P2	B1P3	B1P4	B1P5	B1P6	B1P7	B1P8	B1P9
B2P1	B2P2	B2P3	B2P4	B2P5	B2P6	B2P7	B2P8	B2P9
B3P1	B3P2	B3P3	B3P4	B3P5	B3P6	B3P7	B3P8	B3P9
B4P1	B4P2	B4P3	B4P4	B4P5	B4P6	B4P7	B4P8	B4P9
B5P1	B5P2	B5P3	B5P4	B5P5	B5P6	B5P7	B5P8	B5P9
B6P1	B6P2	B6P3	B6P4	B6P5	B6P6	B6P7	B6P8	B6P9
B7P1	B7P2	B7P3	B7P4	B7P5	B7P6	B7P7	B7P8	B7P9

Okay, that was damn impressive and spooky as well.

Exercise: Can you show $\mathbb{Q}$ can also be fit into the hotel by similar argument?
(Reward: 5 🍬🍬🍬🍬🍬)

The hotel runs out of rooms!

But, now arrives a bus with infinitely many people. Each person has a name, a string consisting of two letters A, B. For example, one of them is named $ABABAAAABBBBBABABAB\cdots$ and another is named $AAAAAAAAAAAAAAAA\cdots$. They all want a room. This time the manager takes some time. After 5 minutes, the manager says. "Sorry, I cannot get rooms for all of you, I shall explain." Suppose you could assign rooms to all of them in some order. Then,

A	B	A	A	A	B	$\cdots$
B	B	B	A	A	A	$\cdots$
A	A	B	A	B	B	$\cdots$
A	A	A	A	A	A	$\cdots$
B	A	A	B	A	B	$\cdots$
A	B	A	B	B	A	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Then the person named $BAABBB\cdots$ is not in the list. Can you see why ?

So, wait. What did we discover?

- Some infinities are bigger than others.
- There are different kinds of infinities.

This was the brainchild of Georg Cantor.

" No one shall expel us from the paradise which Cantor has created for us"-Hilbert.

$\aleph_0, \aleph_1$ and Continuum hypothesis

So, the first kind of infinity that we dealt it. The ones we could fit into the Hotel are called "countable". The size of such infinities is denoted by $\aleph_0$. The second kind is an "uncountable" infinity.

Continuum hypothesis: Is there an infinity bigger than $\aleph_0$ but smaller than $\aleph_1$ OR we wish to show that the real numbers are exactly the ones with size $\aleph_1$

$$2^{\aleph_0} = \aleph_1$$

The only result we have is due to **Paul Cohen** (Fields medal for this work in 1966): The Continuum hypothesis is "provably unsolvable". That is, none of the mathematical structures we know of can be used to decide whether the hypothesis is right or wrong.

📚 Recommendations

- Cantor's diagonalisation, Power sets
- Axiom of choice
- Gödel Incompleteness theorem