The Tragedy/Gift of Mathematics
Gödel's Incompleteness Theorem(s)

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Examples of sets

Have we seen any infinite collections before this ?

- $\mathbb{N} = \{1, 2, \ldots, \}$
- $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$. Like $\frac{1}{2}, \frac{2}{3}$ etc.
- $\mathbb{R}$ like $\sqrt{2}, \pi, e, \pi^e, e^\pi, 2^\pi, etc.$

These sets are clearly infinite. Think for a moment why (convince yourselves).

Cantor's Paradise

Clearly $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. We can show that there are as many naturals as there are integers. Also, there are as many rationals as naturals. BUT, one can show that there are more reals than rationals (not so obvious; try to see why).

Fatal Flaws of (Naïve) Set Theory

First Example: Suppose $\mathcal{S}$ is the set that contains all sets but not itself. We want to ask whether the set $\mathcal{S}$ lies in the set $\mathcal{S}$. Well,

- Suppose $\mathcal{S}$ contains itself, then it contradicts its construction (look at how it was defined).
- Suppose $\mathcal{S}$ does not contain itself, then it has to be inside $\mathcal{S}$ (again by definition).

So we are in a loop with no definite answer!. Or more precisely, the problem is **undecidable**.

Consider a slightly more *worldly problem*. Suppose Bangalore had this law: The barber cuts hair of only those people who do not cut their own hair. Then does the barber cut his own hair? Again,

- If he does not cut his own hair, then by definition the barber has to cut his hair, but he cannot do that.
- If he cuts his own hair, then he would have violated the condition.

Again an **undecidable** problem. Oh God! Really frustrating!

> A fun thought. Have we been learning the wrong mathematics all this time?

Axioms

Axioms are basically rules that you start with, they are the basic building blocks you deduce results from. Let us see one of the most basic Axiom Sets:

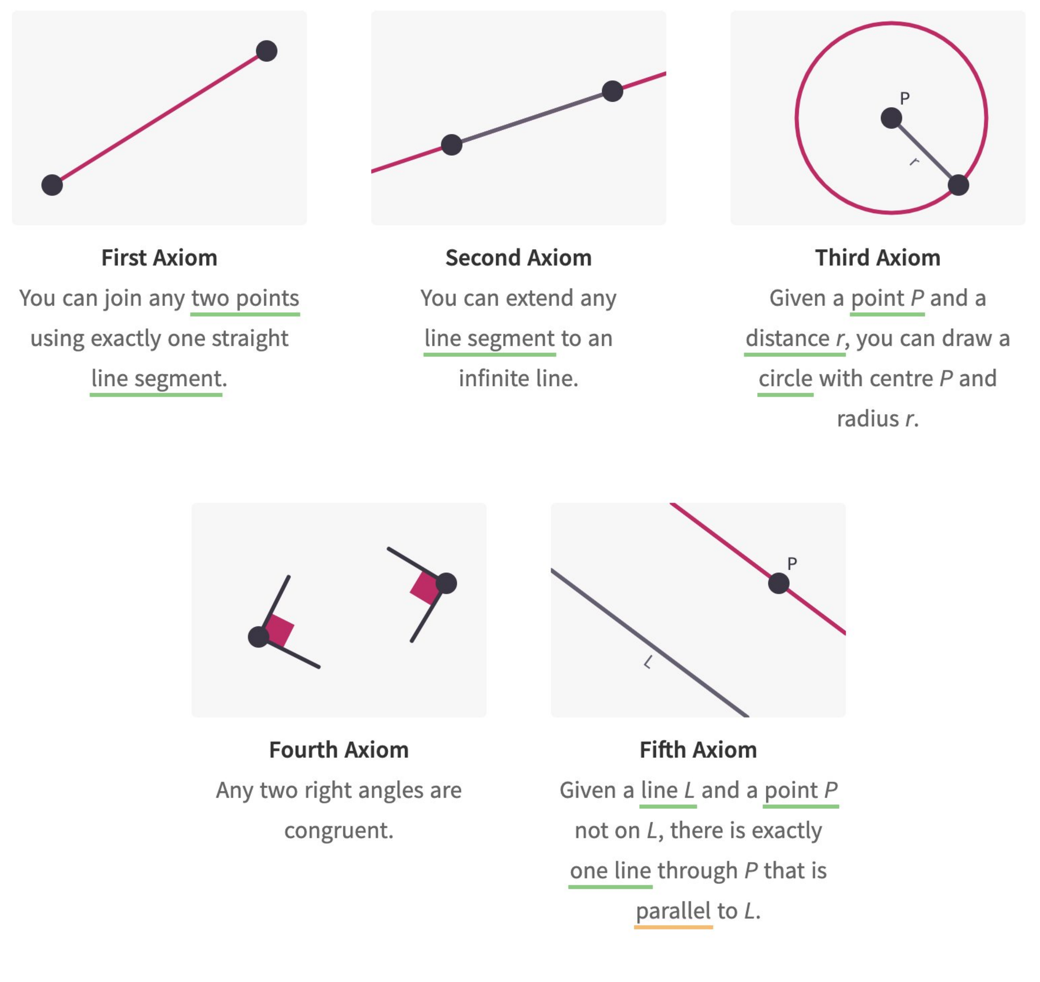

Figure 1. Euclid's axioms

Hilbert's Problems

Figure 2. David Hilbert

Hilbert posed the following problems about mathematics itself:

1. **Completeness:** We want to know whether every true statement is provable.
2. **Consistency:** Here, we wish to know whether every system is free of false statements.
3. **Decidability:** We ask if there is an algorithm which determines whether a statement follows from the axioms.

Hilbert believed the answers to all the above were positive.

Gödel enters the scene

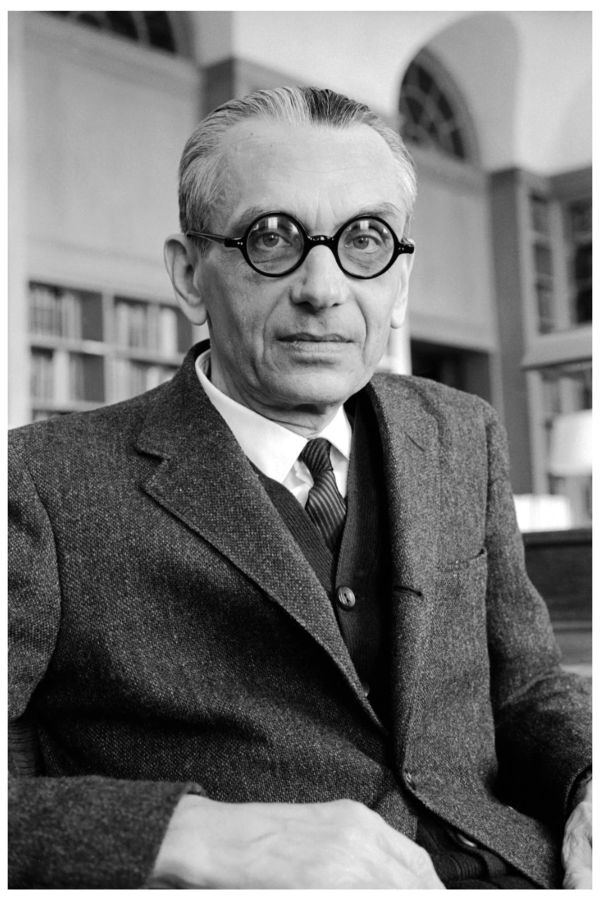

Figure 3. Kurt Gödelabh

Gödel basically addresses the first two questions. He proves:

1. First Incompleteness Theorem Truth and provability are NOT the same. There will always be true statements that cannot be proven.
2. Second Incompleteness Theorem Any consistent system of mathematics cannot prove its own consistency. This is to say that there can be internal contradictions in a mathematical system.

The third problem of DECIDABILITY

This is where we see the birth of modern day computers. There is an intimate relation of Turing machines and Halting Problem to the Decidability Question.

Upshot: Turing showed that there is NO algorithm to show if a statement can be deduced from the axioms.

Some (Un)solved Problems in Mathematics

Consider the theorem of Pythagoras:

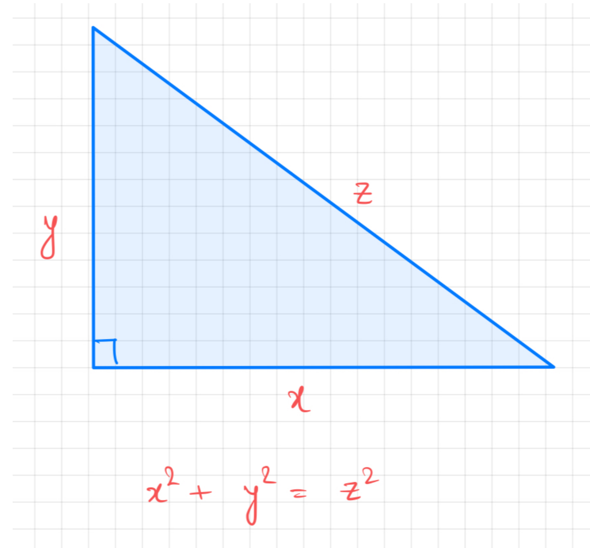

Figure 4. Pythagoras' theorem

Some Pythagorean triples are $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$. It is natural to ask if there are solutions to

$$x^n + y^n = z^n$$

over the integers $\mathbb{Z}$ and $n > 2$.

Fermat famously quipped that he had a proof of the theorem but the *margin was too small to contain it*. After 300 years, Sir Andrew Wiles finally proved it in early 1990s using highly complicated methods from the theory of modular forms and elliptic curves.

Another famous problem is the **Riemann hypothesis**: Consider the function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots$$

defined for $\mathrm{Re}(s) > 1$. Riemann conjectures that all the zeros of the function are on the line $\mathrm{Re}(s) = 1/2$.

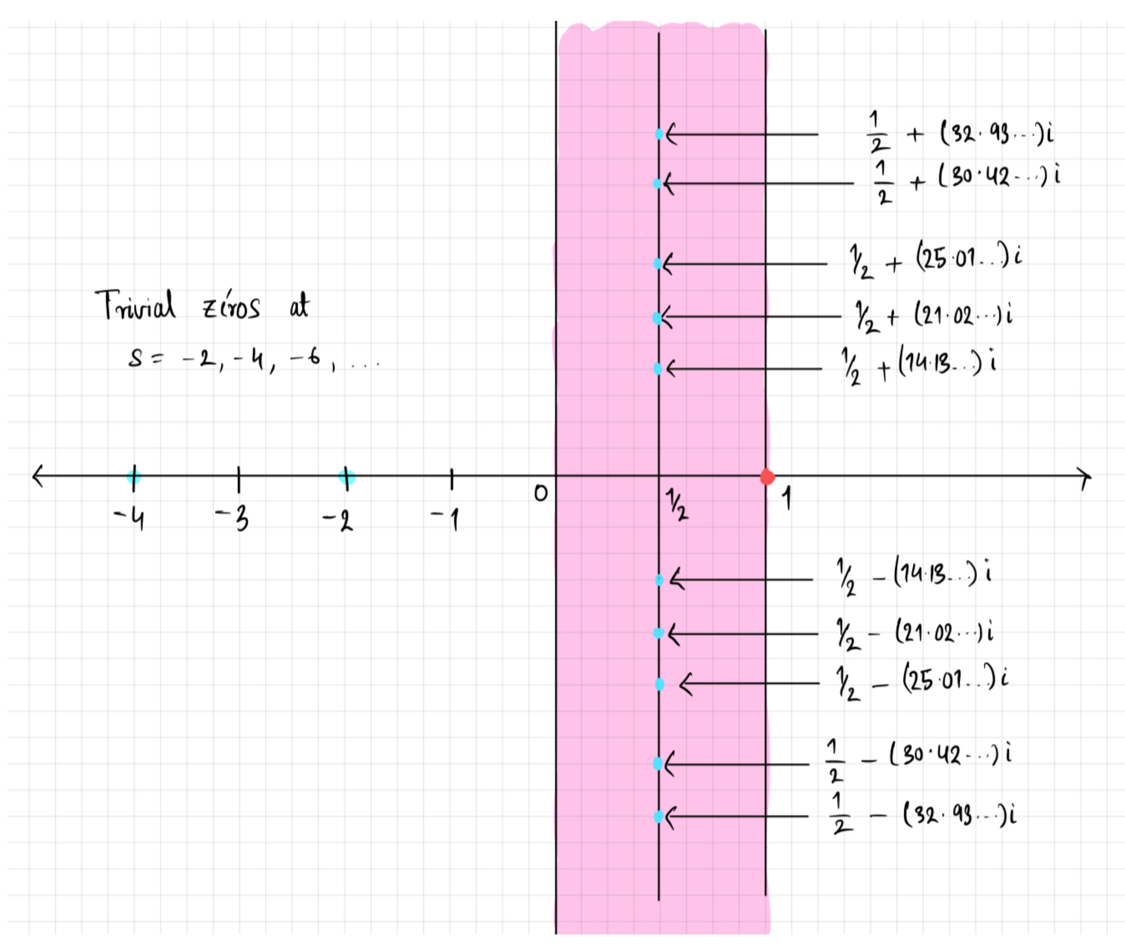

Figure 5. Zeros of the $\zeta(s)$ -function

📚 Recommendations

- Cantor's diagonalisation, Power sets
- Axiom of choice
- Clay Mathematics Institute description of Millenium Problems

Bounty Questions (only for students in class 11 and below)

1. If you can recite π to 7 decimal points you get 2 🍬🍬 and 3 🍬🍬🍬 if you can recite e upto 7 decimal points.
   14 March is π day!
2. Can you prove that the sum
   $$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$
   diverges? 🍬🍬🍬🍬🍬🍬🍬