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Tate's thesis: Fourier Analysis on Number Fields

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Outline

1. History of zeta and L-functions
2. Ring of adèles and idèles
3. Tate's proof of functional equation

Riemann's zeta function and Dirichlet's L-function

1. The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is a priori defined for $\mathrm{Re}(s) > 1$, with Euler product decomposition:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

2. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ be a Dirichlet character. Assume that the character is extended to all of $\mathbb{Z}$ in the usual manner and also that the character is primitive. Then, we have an associated L-function:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

The sum is defined for $\mathrm{Re}(s) > 0$ to begin with. Also, the sum admits an Euler product decomposition of the form :

$$L(s, \chi) = \prod_{p \nmid N} (1 - \chi(p) p^{-s})^{-1}$$

Dedekind's zeta function and Hecke's L-function

3. If K is a number field (finite extension of $\mathbb{Q}$) with ring of algebraic integers $\mathcal{O}_K$, then we may define a ζ-function attached to K as

$$\zeta_K(s) = \sum_{0 \neq \mathfrak{a}} \frac{1}{\mathcal{N}\mathfrak{a}^s}$$

Unique factorisation of fractional ideals yields the Euler product decomposition:

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathcal{N}\mathfrak{p}^s}\right)^{-1}$$

4. Hecke proved the analytic continuation of and functional equation for Dedekind ζ-functions, and for more general class of L-functions

$$L(s, \chi) = \sum_{\mathfrak{a}, (\mathfrak{a}, \mathfrak{m}) = 1} \frac{\chi(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}$$

where χ is what we now call a Hecke character or equivalently a Größencharactere or an idèle class character of conductor $\mathfrak{m}$.

Adèle ring

To rectify this issue, we define adèle ring using the restricted product topology. So, the adèle ring (denoted by $\mathbb{A}$) is defined as

$$\mathbb{A} = \coprod_{v \in M_{\mathbb{Q}}} (\mathbb{Q}_v, \mathbb{Z}_v) := \{(x_v) \in \prod_{v \in M_K} \mathbb{Q}_v \text{ such that } x_v \in \mathbb{Z}_v \text{ for almost all } v\}$$

- This is a ring with operations

$$(x_\infty, x_2, x_3, \ldots) + (y_\infty, y_2, y_3, \ldots) = (x_\infty + y_\infty, x_2 + y_2, x_3 + y_3, \ldots)$$
$$(x_\infty, x_2, x_3, \ldots) \cdot (y_\infty, y_2, y_3, \ldots) = (x_\infty \cdot y_\infty, x_2 \cdot y_2, x_3 \cdot y_3, \ldots)$$

- This is also a topological ring with basic open sets

$$\prod_v U_v$$

  with U_v open in $\mathbb{Q}_v$ and $U_v = \mathbb{Z}_v$ for almost all v.

- The projection map $x \mapsto x_v$ is continuous, and thus it is finer than the product topology.

Idèle ring

Now, we are also concerned with the group of units of the adèle ring which we call idèles. We define the group (denoted by $\mathbb{A}^\times$) by

$$\mathbb{A}^\times = \coprod_v (\mathbb{Q}_v^\times, \mathbb{Z}_v^\times)$$

- This is a topological group with basic open sets

$$\prod_v U_v$$

  with U_v open in $\mathbb{Q}_v^\times$ and $U_v = \mathbb{Z}_v^\times$ for almost all v.
- This topology is finer than the induced topology.

Properties of adèle and idèle ring

1. $\mathbb{A}$ is locally compact and Hausdorff.
2. $\mathbb{Q} \hookrightarrow \mathbb{A}$ (diagonal embedding) is discrete, and the quotient $\mathbb{A}/\mathbb{Q}$ is compact.
3. $\mathbb{Q}$ acts on $\mathbb{A}$ by right translation. This lets us define a fundamental domain

$$\mathbb{A}/\mathbb{Q} \simeq D := [0,1) \times \prod_p \mathbb{Z}_p$$

OR

$$\mathbb{A} = \bigsqcup_{k \in \mathbb{Q}} (k + D)$$

4. Similarly, we can also embed $\mathbb{Q}^\times$ diagonally into $\mathbb{A}^\times$, and again we can show that this image is discrete and moreover $\mathbb{A}^\times/\mathbb{Q}^\times$ is compact.
5. Again, we can define a fundamental domain for the action of $\mathbb{Q}^\times$ on $\mathbb{A}^\times$ by

$$\mathbb{A}^\times/\mathbb{Q}^\times \simeq E := (0,\infty) \times \prod_p \mathbb{Z}_p$$

OR

$$\mathbb{A}^\times = \bigsqcup_{k \in \mathbb{Q}^\times} kE$$

Local measures

Existence of Haar measure

A locally compact group G has a Haar measure unique upto a positive scalar.

As a consequence, the locally compact group $\mathbb{Q}_v^+$ has a Haar measure.

- Let dx be a Haar measure on $\mathbb{Q}_v^+$.
- Then, $d^\times x = \dfrac{dx}{|x|_v}$ is a Haar measure on $\mathbb{Q}_v^\times$. We also choose this measure such that $\mathrm{vol}(\mathbb{Z}_p) = 1$.

Local characters

For the local field $\mathbb{Q}_v$

- The additive group $\mathbb{Q}_\infty^+ = \mathbb{R}^+$ is naturally isomorphic to its character group S^1, the isomorphism being given by

$$y \mapsto (x \mapsto e^{2\pi i x y})$$

- When we are looking at $\mathbb{Q}_p$, the map $e_p : \mathbb{Q}_p \to S^1$ given by

$$e_p\left(\sum_{n=-N}^{\infty} a_n p^n\right) = \exp\left(2\pi i \sum_{n=-N}^{-1} a_n p^n\right)$$

  has the property that if $\chi : \mathbb{Q}_p \to S^1$ is a group homomorphism then there is exactly one $a \in \mathbb{Q}_p$ such that $\chi(x) = e_p(ax)$

- More compactly, the additive group $\mathbb{Q}_v^+$ is naturally isomorphic to its character group $\widehat{\mathbb{Q}_v^+}$ via the map $\xi \mapsto (\eta \mapsto e^{2\pi i \Lambda(\xi\eta)})$

Fourier transforms

Recall the space $\mathcal{S}(\mathbb{R})$ of all infinite differentiable functions $f : \mathbb{R} \to \mathbb{C}$ such that for any two $m, n \in \mathbb{Z}_{\geq 0}$, the function $x^m f^{(n)}(x)$ is bounded.

For p a prime, a Schwartz-Bruhat function is a function $f : \mathbb{Q}_p \to \mathbb{C}$ that is locally constant with a compact support. We denote the space of such functions by $\mathcal{S}(\mathbb{Q}_p)$. Now, we can define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{Q}_p)$ for $p \leq \infty$. Let,

$$\widehat{f}(y) = \int_{\mathbb{Q}_p} f(x) e_p(-xy) dx$$

Fourier inversion formula

There is an unique measure on the character group ($d\chi$ the Plancherel measure) such that the inversion formula

$$f(x) = \int_{\widehat{\mathbb{Q}}_p} \widehat{f}(\chi)\chi(x) d\chi = \widehat{\widehat{f}}(-x)$$

for "good" enough f.

Archimedean local zeta function

Recall that $f_\infty(x_\infty) = e^{-\pi x_\infty^2}$ is self dual, i.e., $\widehat{f_\infty} = f_\infty$. The Archimedean local ζ-function is :

$$\begin{aligned}\zeta(f_\infty, s) &:= \int_{\mathbb{R}^\times} f_\infty(x_\infty)|x_\infty|_\infty^s d^\times x_\infty \\ &= 2\int_0^\infty e^{-\pi x_\infty^2}|x_\infty|_\infty^s \frac{dx_\infty}{|x_\infty|_\infty} \\ &= \pi^{-s/2}\int_0^\infty e^{-t} t^{s/2}\frac{dt}{t} \\ &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\end{aligned}$$

p-adic local zeta function

Recall that $f_p = \mathbf{1}_{\mathbb{Z}_p}$ is self dual, i.e., $\widehat{f_p} = f_p$. The non-Archimedean local ζ-function is :

$$\begin{aligned}
\zeta(f_p, s) &:= \int_{\mathbb{Q}_p^\times} f_p(x_p)|x_p|_p^s d^\times x_p \\
&= \int_{\mathbb{Z}_p^\times} |x_p|_p^s \frac{p}{p-1} \frac{dx_p}{|x_p|_p} \\
&= \frac{p}{p-1} \sum_{k \geq 0} \int_{p^k \mathbb{Z}_p} |x_p|_p^{s-1} dx_p \\
&= \frac{p}{p-1} \sum_{k \geq 0} p^{k(s-1)} \mathrm{vol}(p^k \mathbb{Z}_p) \\
&= \frac{p}{p-1} \sum_{k \geq 0} p^{k(s-1)} \frac{p-1}{p} p^{-k} \\
&= \sum_{k \geq 0} p^{-ks} \\
&= (1-p^{-s})^{-1}
\end{aligned}$$

Adèlic measure

The Schwartz class of functions $\mathcal{S}(\mathbb{A}_{\text{fin}})$ is the space of all functions $g : \mathbb{A}_{\text{fin}} \to \mathbb{C}$ which are locally constant and have compact support.
By $\mathcal{S}(\mathbb{A})$, we denote the space of all functions of the form

$$g(x) = \sum_{j=1}^{n} h_j(x_{fin}) r_j(x_\infty)$$

where $h_j \in \mathcal{S}(\mathbb{A}_{\text{fin}})$ and $r_j \in \mathcal{S}(\mathbb{R})$.
It can be shown that every function $f \in \mathcal{S}(\mathbb{A})$ can be written as a finite sum of functions of the form

$$\prod_{v \in M_{\mathbb{Q}}} f_v$$

with $f_v \in \mathcal{S}(\mathbb{Q}_v)$ and $f_p = \mathbf{1}_{\mathbb{Z}_p}$ for almost all p.
Since $\mathbb{A}$ is a locally compact group, it has a Haar measure which can be normalised such that for every integrable simple function $f = \prod_p f_p$, one has the product formula

$$\int_{\mathbb{A}} f(x)dx = \prod_p \int_{\mathbb{Q}_p} f_p(x_p)dx_p$$

Adélic characters and Fourier analysis

Theorem

For every $x \in \mathbb{A}$, the product

$$e(x) = \prod_{v \in M_{\mathbb{Q}}} e_p(x_p)$$

has only finitely many factors not equal to 1, i.e., the product is finite. The ensuing map $e : \mathbb{A} \to S^1$ is a character.
Moreover, for every character $\chi : \mathbb{A} \to S^1$, there exists an uniquely determined $a \in \mathbb{A}$ with $\chi(x) = e(ax)$. The map $\chi \mapsto a$ is the isomorphism of $\widehat{\mathbb{A}}$ to $\mathbb{A}$.

For $f \in \mathcal{S}(\mathbb{A})$, we define the Fourier transform by:

$$\widehat{f}(y) := \int_{\mathbb{A}} f(x) e(-xy) dx$$

Adèlic Fourier analysis

Again, for "good" enough functions, we have the Fourier inversion formula:

Theorem

For every function $f \in \mathcal{S}(\mathbb{A})$ of the form $f = \prod_{v \in M_{\mathbb{Q}}} f_v$ with $f_v \in \mathcal{S}(\mathbb{Q}_v)$, one has

$$\widehat{f} = \prod_{v \in M_{\mathbb{Q}}} \widehat{f}_v$$

For $f \in \mathcal{S}(\mathbb{A})$ one has $\widehat{f} \in \mathcal{S}(\mathbb{A})$ and the inversion formula for the Fourier transformation is

$$\widehat{\widehat{f}}(x) = f(-x)$$

Adèlic zeta function

Let $f = \prod_v f_v$ be the self-dual function. Then,

$$\begin{aligned}\zeta(f,s) &:= \int_{\mathbb{A}^\times} f(x)|x|^s d^\times x \\ &= \prod_v \int_{\mathbb{Q}_v^\times} f_v(x_v)|x_v|^s d^\times x_v \\ &= \pi^{-s/2}\Gamma(s/2)\prod_p (1-p^{-s})^{-1}\end{aligned}$$

The adèlic zeta function just becomes the usual completed zeta function.

Adèlic Poisson Summation formula

Classical version

For every $f \in \mathcal{S}(\mathbb{R})$, one has

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)$$

Adèlic version

For every $f \in \mathcal{S}(\mathbb{A})$, one has

$$\sum_{q\in\mathbb{Q}} f(q) = \sum_{q\in\mathbb{Q}} \widehat{f}(q)$$

where the series are both absolutely convergent.

Proof of Poisson summation formula

Proof.

Let $F(x) := \sum_{q\in\mathbb{Q}} f(q+x)$ and ψ be a $\mathbb{Q}$-invariant character of $\mathbb{A}$. Then,

$$\begin{aligned}\widehat{F}(k) &= \int_{\mathbb{A}/\mathbb{Q}\simeq D} \left(\sum_{q\in\mathbb{Q}} f(x+q)\right)\psi(kx)dx \\ &= \sum_{q\in\mathbb{Q}} \int_D f(x+q)\psi(kx)dx \\ &= \sum_{q\in\mathbb{Q}} \int_{D+q} f(x)\psi(k(x-q))dx \\ &= \int_{\mathbb{A}} f(x)\psi(kx)dx \\ &= \widehat{f}(k)\end{aligned}$$

Now, $F(x) = \sum_{q\in\mathbb{Q}} \widehat{f}(q)\overline{\psi}(qx)$. So, evaluate at $x=0$ to obtain the result. □

Adèlic theta function and analytic continuation

Define $\Theta(x) = \sum_{q \in \mathbb{Q}} f(qx)$. By applying adélic Poisson summation formula, one has

$$\Theta(x) = \frac{1}{|x|}\Theta\left(\frac{1}{x}\right)$$

By slicing the integral into $\mathbb{Q}^\times$ classes, one gets

$$\begin{aligned}\zeta(f,s) &= \int_{\mathbb{A}^\times} f(x)|x|^s d^\times x \\ &= \int_{\mathbb{A}^\times/\mathbb{Q}^\times \simeq E} \sum_{q \in Q} f(qx)|qx|^s d^\times x\end{aligned}$$

After some manipulation, one obtains

$$\zeta(f,s) + f(0)\int_{\mathbb{A}^\times} |x|^s d^\times x = \zeta(f,1-s) + f(0)\int_{\mathbb{A}^\times} |x|^{1-s} d^\times x$$

This is the desired analytic continuation.

Thank you! Available for questions

Picture of Letter from Iwasawa to Dieudonne, Kenichi Iwasawa, and John Tate.

April 8, 1952

Professor J. Dieudonne
26 Rue Saint-Michel, Nancy
France

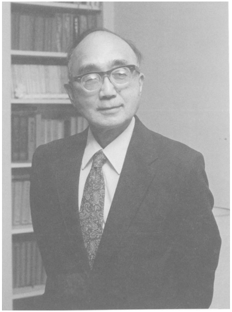
Dear Professor Dieudonne:

A few days ago, I received a letter from Professor A. Weil, asking me to send you a copy of a letter I wrote him the other day and to give you a brief account of my result on L-functions. I, therefore, enclose here a copy of that letter and write an outline of my idea on L-functions.

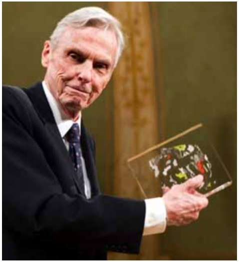
Let k be a finite algebraic number field, J the idele group of k, topologized as in a recent paper of Weil. J is a locally compact abelian group containing the principal idele group P as a discrete subgroup. We denote by J_0 the subgroup of J consisting of ideles $\mathfrak{a} = (a_p)$ such that $a_p = 1$ for all infinite (i.e. archimedean) primes P. We call J_0 the finite part of J and define the infinite part J_∞ similarly, so that we have

$$J = J_0 \times J_\infty, \quad \mathfrak{a} = \mathfrak{a}_0\mathfrak{a}_\infty, \quad \mathfrak{a}_0 \in J_0, \quad \mathfrak{a}_\infty \in J_\infty.$$

We also denote by U the compact subgroup of J consisting of ideles $\mathfrak{a} = (a_p)$ such that the absolute value $|a_p|_p = 1$ for every prime P. $U_0 = U \cap J_0$ is then an open, compact subgroup of J_0 and J_0/U_0 is canonically isomorphic to the ideal group I of k. According to Artin-Whaples, we can choose the absolute values $|a_p|_p$ so that the volume function $V(\mathfrak{a}) = \Pi_p |a_p|_p$ $(\mathfrak{a} = (a_p))$ has the value 1 at every principal idele $\alpha \in P$ (the product formula) and that $V(\mathfrak{a}_0)^{-1}$ is equal to the absolute norm $N(\tilde{\mathfrak{a}}_0)$ of the ideal $\tilde{\mathfrak{a}}_0$, which corresponds to $\mathfrak{a}_0$ by the above isomorphism between J_0/U_0 and I.



K. Iwasawa
Princeton, 1986



Receiving the Abel Prize in 2010

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