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Kubota-Leopoldt p-adic L-functions and p-adic modular forms

Department of Mathematics, IISc Bangalore, India.

Undergraduate Summer Project Presentation,
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Outline

References I

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Mellin transform

Lemma and definition

If $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ is a C^∞-function on $\mathbb{R}_{\geq 0}$, such that f and all its derivatives decay exponentially at infinity, and

$$L(f,s) := \frac{1}{\Gamma(s)} \int_0^\infty f(t) t^{s-1} dt, \; s \in \mathbb{C} \tag{1}$$

Then, $L(f,s)$ converges to a holomorphic function for $\mathrm{Re}(s) > 0$, has an analytic continuation to the entire complex plane, and

$$L(f,-n) = (-1)^n \frac{d^n}{dt^n} f(0) \tag{2}$$

$L(f,s)$ as defined above is called the Mellin transform of f.

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Bernoulli numbers and ζ-function

Consider the power series expansion of the function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{3}$$

B_n are called Bernoulli numbers, and $B_n \in \mathbb{Q}$ with values:

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}$$

Note that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} f(t) t^{s-1} \frac{dt}{t} = \frac{1}{s-1} L(f, s-1) \tag{4}$$

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continued

Applying lemma to $f(t) = \dfrac{t}{e^t - 1}$ we have

Theorem

$\zeta(s)$ has a meromorphic continuation to all of $\mathbb{C}$. It is holomorphic everywhere except for a simple pole at $s = 1$ with residue $L(f, 0) = B_0 = 1$. If $n \geq 2$, then

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \in \mathbb{Q}$$

Kummer's congruence and Kummer's theorem

Kummer

If p does not divide the numerator of $\zeta(-3), \zeta(-5), \ldots, \zeta(2-p)$, then $p \nmid \# Cl(\mathbb{Q}(\mu_p))$.

The primes p that do not divide the class number of $\mathbb{Q}(\mu_p)$ are known as regular primes and irregular otherwise. It is know that there are infinitely many irregular primes but the infinitude of of regular primes is still an open problem.

Kummer's congruences

Let $a \geq 2$ be prime to p. Let $k \geq 1$. If $n_1, n_2 \geq k$ such that $n_1 \equiv n_2 \pmod{(p-1)p^{k-1}}$, then

$$(1 - a^{1+n_1})\zeta(-n_1) \equiv (1 - a^{1+n_2})\zeta(-n_2) \pmod{p^k}$$

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p-adic Banach spaces

Definition

A p-adic Banach space B is a $\mathbb{Q}_p$ vector space with a lattice B^0 ($\mathbb{Z}_p$-module) separated and complete for the p-adic topology, i.e.,

$$B^0 \simeq \varprojlim_n B^0/p^n B^0$$

For each $x \in B$, there is a $n \in \mathbb{N}$ such that $x \in p^n B^0$. We define

$$v_B(x) = \sup_{n \in \mathbb{N} \cup \{\infty\}} \{n : x \in p^n B^0\}$$

It has the properties of a valuation, i.e.,

- $v_B(x+y) \geq \min\{v_B(x), v_B(y)\}$
- $v_B(\lambda x) = v_p(\lambda) + v_B(x)$ for $\lambda \in \mathbb{Q}_p$

Then, $||x||_B = p^{-v_B(x)}$ is a norm on B, and B^0 is the unit ball.

Examples

- $B = \mathbb{C}_p$ with $B^0 = \mathcal{O}_{\mathbb{C}_p}$
- $B = \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Q}_p)$ with $B^0 = \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Z}_p)$

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Continuous functions on $\mathbb{Z}_p$

Consider the binomial coefficient:

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$$

It is easy to see that $\binom{x}{n} \in \mathbb{Z}_p$ (Hint: Look at what happens for $x \in \mathbb{N}$ and observe that $\mathbb{N}$ is dense in $\mathbb{Z}_p$).
For $f \in \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Q}_p)$, set

$$f^{[0]} = f,\ f^{[k+1]}(x) = f^{[k]}(x+1) - f^{[k]}(x)$$

and Mahler's coefficient is

$$a_n(f) = f^{[n]}(0)$$

Hence,

$$f^{[n]}(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x+n-i)$$

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Mahler's theorem

Theorem

If $f \in \mathcal{C}^0(\mathbb{Z}_p, \mathbb{C}_p)$, then

1. $\lim_{n \to \infty} v_p(a_n(f)) = +\infty$
2. $\forall x \in \mathbb{Z}_p, f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n}$
3. $v_{\mathcal{C}^0}(f) = \inf v_p(a_n(f))$

The main thing to note is the second point. Any continuous function f can be written as linear combination of binomial coefficients with the constants tending to 0 (p-adically) as n increases.

Locally constant functions

Say $z \in \mathbb{C}_p$ such that $v_p(z-1) > 0$. Consider

$$f_z(x) = \sum_{n=0}^{\infty} \binom{x}{n} (z-1)^n \in \mathcal{C}^0(\mathbb{Z}_p, \mathbb{C}_p)$$

Since for $k \in \mathbb{N}$, $f_z(k) = z^k$, therefore $f_z(x) = z^x$, moreover $z^{x+y} = z^x z^y$.
If z is a primitive p-th root of 1, then

$$v_p(z-1) = \frac{1}{(p-1)p^{n-1}} > 0$$

Note that $z^{x+p^n} = z^x$ for all x, then z^x is locally constant. The characteristic function of $k + p^n\mathbb{Z}_p$ is given by

$$\begin{aligned}
\mathbf{1}_{k+p^n\mathbb{Z}_p}(x) &= \mathbf{1}_{p^n\mathbb{Z}_p}(x-k) \\
&= \frac{1}{p^n} \sum_{z^{p^n}=1} z^{x-k} \\
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Amice Transform

Definition

The Amice transform of a measure μ is defined to be the map:

$$\mu \mapsto A_\mu(T) = \int_{\mathbb{Z}_p} (1+T)^x \mu(x) = \sum_{n=0}^{\infty} T^n \int_{\mathbb{Z}_p} \binom{x}{n} \mu$$

Theorem

The map $\mu \mapsto A_\mu$ is an isometry from $\mathcal{D}^0(\mathbb{Z}_p, \mathbb{Q}_p)$ to the set $\{\sum_{n=0}^{\infty} b_n T^n, \mathbb{Q}_p \ni b_n \text{ bounded }\}$ with the valuation $v\left(\sum_{n=0}^{\infty} b_n T^n\right) = \inf_{n \in \mathbb{N}} v_p(b_n)$

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Properties of Amice Transform

- *Multiplication of a measure by continuous function*. For $\mu \in \mathcal{D}^0, f \in \mathcal{C}^0$, we define the measure $f\mu$ by

$$\int_{\mathbb{Z}_p} g \cdot f\mu = \int_{\mathbb{Z}_p} f(x)g(x)\mu(x)$$

for all $g \in \mathcal{C}^0$.

- If $f(x) = x$, then

$$A_{x\mu}(T) = (1+T)\frac{d}{dT}A_\mu$$

- If $f(x) = z^x$, then

$$A_{z^x\mu}(T) = A_\mu((1+T)z - 1)$$

- *Actions of* φ, ψ. For $\mu \in \mathcal{D}^0$, we define the action of φ on μ by

$$\int_{\mathbb{Z}_p} f(x)\varphi(\mu) = \int_{\mathbb{Z}_p} f(px)\mu(x)$$

Hence, $A_{\varphi(\mu)}(T) = A_\mu((1+T)^p - 1) = \varphi(A_{\mu(T)})$

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- If $f(x) = z^x$, then

$$A_{z^x\mu}(T) = A_\mu((1+T)z - 1)$$

- *Actions of φ, ψ.* For $\mu \in \mathcal{D}^0$, we define the action of φ on μ by

$$\int_{\mathbb{Z}_p} f(x)\varphi(\mu) = \int_{\mathbb{Z}_p} f(px)\mu(x)$$

  Hence, $A_{\varphi(\mu)}(T) = A_\mu((1+T)^p - 1) = \varphi(A_{\mu(T)})$

Properties continued

- We define the action of ψ by

$$\int_{\mathbb{Z}_p} f(x)\psi(\mu) = \int_{\mathbb{Z}_p} f(x/p)\mu(x)$$

Therefore, $A_{\psi(\mu)} = \psi(A_\mu)$. Additionally,

  - $\psi \circ \varphi = \mathrm{Id}$
  - $\mathrm{Res}_{\mathbb{Z}_p^\times}(\mu) = (1 - \varphi\psi)\mu$

- *Convolution of measures*. If λ, μ are two measures, then their convolution $\lambda * \mu$ is defined by

$$\int_{\mathbb{Z}_p} f(x)\lambda * \mu = \int_{\mathbb{Z}_p} \left(\int_{\mathbb{Z}_p} f(x+y)\mu(x) \right) \lambda(y)$$

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Kummer's congruence proof

Lemma

For $a \in \mathbb{Z}_p^\times$, there exists a measure λ_a such that

$$A_{\lambda_a} = \int_{\mathbb{Z}_p} (1+T)^x \lambda_a = \frac{1}{T} - \frac{a}{(1+T)^a - 1}$$

Proposition

For every $n \in \mathbb{N}$,

$$\int_{\mathbb{Z}_p} x^n \lambda_a = (-1)^n (1 - a^{1+n}) \zeta(-n)$$

Corollary

For $a \in \mathbb{Z}_p^\times$, $k \geq 1, n_1, n_2 \geq k, n_1 \equiv n_2 \pmod{p^{k-1}(p-1)}$, we have

$$v_p(1 - a^{1+n_1})\zeta(-n_1) - (1 - a^{1+n_2})\zeta(-n_2) \geq k$$

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Restriction to $\mathbb{Z}_p^\times$

1. $\psi(1/T) = 1/T$
2. $\psi(\lambda_a) = \lambda_a$
3. $\mathrm{Res}_{\mathbb{Z}_p^\times}(\lambda_a) = (1 - \varphi\psi)\lambda_a = (1 - \varphi)\lambda_a$
4. $\int_{\mathbb{Z}_p^\times} x^n \lambda_a = \int_{\mathbb{Z}_p} x^n (1 - \varphi)\lambda_a = (-1)^n (1 - a^{n+1})(1 - p^n)\zeta(-n)$

Leopoldt's Γ-transform

Teichmüller character

For $x \in \mathbb{Z}_p^\times, \omega(x) = \lim_{n\to\infty} x^{p^n}$

Key point: Every element $x \in \mathbb{Z}_p^\times$ can be uniquely written as $x = \omega(x)\langle x\rangle$. Moreover, $\omega(xy) = \omega(x)\omega(y)$ and consequently $\langle xy\rangle = \langle x\rangle\langle y\rangle$

Proposition

If λ is a measure on $\mathbb{Z}_p^\times$, $u = 1 + 2p$, then there exists a measure $\Gamma_\lambda^{(i)}$ on $\mathbb{Z}_p$ (called the Leopoldt transform) such that

$$A_{\Gamma_\lambda^{(i)}}(u^s - 1) = \int_{\mathbb{Z}_p^\times} \omega(x)^i \langle x\rangle^s \lambda(x)$$

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p-adic ζ-functions

Definition

For $i \in \mathbb{Z}/\phi(2p)\mathbb{Z}$, and $a \in \mathbb{Z}_p^\times$ such that $\langle a \rangle \neq 1$, we define a function on $\mathbb{Z}_p$ as

$$\zeta_{p,i} = \frac{1}{1-\omega(a)^{1-i}\langle a \rangle^{1-s}} A_{\Gamma_{\lambda_a}^{(-i)}}(u^{-s}-1)$$

More explicity,

$$\zeta_{p,i} = \frac{1}{1-\omega(a)^{1-i}\langle a \rangle^{1-s}} \int_{\mathbb{Z}_p^\times} \omega(x)^{-i}\langle x \rangle^{-s}\lambda_a(x)$$

Theorem

For $i \in \mathbb{Z}/\phi(2p)\mathbb{Z}$, and $a \in \mathbb{Z}_p^\times$ such that $\langle a \rangle \neq 1$, there exists an unique function $\zeta_{p,i}$, analytic on $\mathbb{Z}_p$ if $i \neq 1$, and $(s-1)\zeta_{p,1}(s)$ is analytic on $\mathbb{Z}_p$, such that

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if $n \equiv -1 \pmod{p-1}, n \in \mathbb{N}$

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p-adic L-functions for Dirichlet characters

Definition

Let $\chi = \theta\eta$ be a Dirichlet character, where η has conductor D prime to p and θ has conductor power of p. We define

$$L_p(\chi, s) := \int_{\mathbb{Z}_p^\times} \theta\omega^{-1}(x)\langle x\rangle^{-s}\mu_\eta$$

Note that

$$\zeta_{p,i}(s) = L_p(\omega^i, s)$$

Therefore for arbitrary $k > 0$, we have

$$\zeta_{p,i}(1-k) = (1 - \omega^{i-k}(p)p^{k-1})L(\omega^{i-k}, 1-k)$$

Theorem

For all $k > 0$, one has

$$L_p(\chi, 1-k) = (1 - \chi\omega^{-k}(p)p^{k-1})L(\chi\omega^{-k}, 1-k)$$

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p-adic Eisenstein series

The **Iwasawa algebra** $\Lambda(\mathbb{Z}_p)$ is the space of all $L/\mathbb{Q}_p$-valued measures on $\mathbb{Z}_p$ (defined as the dual $\mathrm{Hom}_{cts}(\mathcal{C}(\mathbb{Z}_p, L), L)$ equipped with strong topology)

Theorem

The Amice transform gives an $\mathcal{O}_L$-algebra isomorphism

$$\Lambda(\mathbb{Z}_p) \xrightarrow{\sim} \mathcal{O}_L[[T]]$$

Recall that for $k \geq 4$ even integers, we have

$$E_k(z) := \frac{\zeta(1-k)}{2} + \sum_{n\geq 1} \sigma_{k-1}(n) q^n$$

Definition

The p-stabilisation of E_k is defined to be

$$E_k^{(p)}(z) := E_k(z) - p^{k-1} E_k(pz)$$

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Note that

$$E_k^{(p)}(z) := \frac{(1-p^{k-1})\zeta(1-k)}{2} + \sum_{n\geq 1} \sigma_{k-1}^p(n) q^n$$

where

$$\sigma_{k-1}^p(n) = \sum_{d|n, p\nmid d} d^{k-1}$$

Also, $E_k^{(p)}(z)$ is a modular form of weight k and level $\Gamma_0(p)$

Theorem

There exists a power series

$$\mathcal{E}(z) = \sum_{n=0}^{\infty} a_n q^n$$

such that $a_n \in \Lambda(\mathbb{Z}_p^\times)$ for all $n \geq 1$, a_0 is a pseudo-measure and for all $k \geq 4$ and even, we have

$$\int_{\mathbb{Z}_p^\times} x^{k-1}\mathcal{E}(z) := \left(\int_{\mathbb{Z}_p^\times} x^{k-1} a_n\right) q^n = E_k^{(p)}(z)$$

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It's over

Thank you! Always unsure what to write on this slide.