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BACHELOR THESIS

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Tate’s thesis: Fourier analysis on number fields

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Contents

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Introduction

The Riemann-zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

defined for $\text{Re}(s) > 1$, can be extended meromorphically to other values of s by analytic continuation and follows the functional equation [Lan94] [Dav80]

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) \tag{2}$$

Riemann's proof crucially depends on the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(at) = \frac{1}{|t|} \sum_{n\in\mathbb{Z}} \widehat{f}(a/t) \tag{3}$$

where $\widehat{f}$ is the Fourier transform on the reals $\widehat{f}(\xi) := \int_{\mathbb{R}} \exp(-2\pi i\xi x) f(x) dx$ where $\exp(x) = 2\pi ix$ is the standard character from $\mathbb{R}$ to S^1. Applying this formula to the Gaussian function $\exp(\pi|x|^2)$ which is its own (additive) Fourier transform, and then applying Mellin transform (multiplicative Fourier transform), one obtains the functional equation (2). An important thing to notice in the proof is how the additive and multiplicative Fourier transform combine to give the functional equation.

In his PhD thesis [Cas+76] (independently found by Iwasawa [Iwa]), Tate was able to extend this idea to the adéle ring of a number field. He was able to develop Poisson summation formula in this setting 41, apply this formula to the adélic Gauss function which is its own Fourier transform and then apply adélic Mellin transform to obtain a functional equation 27 of the type we saw before.

The significance of this thesis lies in the fact that the theory developed in the process allows one to seamlessly generalise the functional equation to Dedekind zeta functions and Hecke L-functions, thereby giving us a large class of functions whose analytic continuation and functional equation is readily available.

Chapter 1 deals with the local theory. We construct additive and multiplicative characters for local fields. We also construct the additive and multiplicative Haar measure on the local fields. After that we define the local zeta integrals whose analytic continuation and functional equation is the main result of this chapter. In the euler product

$$\left(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right)\prod_{p}(1-p^{-s})^{-1}$$

Each of the term in the product will arise as the local zeta integral defined in this chapter.

Chapter 2 is concerned with the global theory. We again construct additive and multiplicative characters. But this time the characters come directly from the local theory. The measures also come directly from the local theory. The highlight of §2.2 is the Poisson summation formula and Riemann-Roch theorem that will come in handy while proving the analytic continuation and functional equation of the global zeta integral as will be seen in §2.4. The preliminaries about adele rings and idele groups is explained in detail in the appendices $C.4, C.5, C.6$.

In chapter 3, we first give an introduction to classical automorphic forms, then look at automorphic forms defined on $GL(n)$, and observe how Tate's thesis fits into the picture as $GL(1)$ case. In §3.3, we will attach L-functions to automorphic forms and talk of its analytic continuation and functional equation. The proof follows Tate's philosophy. We first construct these Whittaker models which is used to conclude the analytic continuation and functional equation of the local L-functions. We then observe that the global L-functions occur as product of local L-functions and the analytic continuation of the local L-functions allows us to analytically continue the global L-function. We also obtain a functional equation for the global L-function in §§3.3.2.

The necessary results on topological groups, Haar measure, analysis on locally compact abelian groups, Algebraic number theory is provided in the appendices.

1. Local Theory

Let K be a number field, $K_\mathfrak{p}$ the completion of K at the prime $\mathfrak{p}$. If $\mathfrak{p}$ is Archimedean, then $K_\mathfrak{p}$ is either $\mathbb{R}$ or $\mathbb{C}$ and if $\mathfrak{p}$ is non-Archimedean, then $K_\mathfrak{p}$ is $\mathfrak{p}$-adic and contains a ring of integers $\mathcal{O}_\mathfrak{p}$ with a single prime ideal $\mathfrak{p} = \langle \varpi \rangle$ with $\mathcal{O}_\mathfrak{p}/\mathfrak{p}$ finite, the cardinality being $N\mathfrak{p}$. $K_\mathfrak{p}$ is complete and hence a local field.

We choose our valuation on $K_\mathfrak{p}$ such that

- $|\alpha|$ is the normal absolute value on $\mathbb{R}$ if $K_\mathfrak{p}$ is real.
- $|\alpha|$ is square the real absolute value if $K_\mathfrak{p}$ is complex.
- $|\alpha| = (N\mathfrak{p})^{-v} = (\mathcal{O}/\alpha\mathcal{O})^{-1}$ with v the valuation of α.

Lemma 1. *A subset $B \subseteq K_\mathfrak{p}$ has compact closure if and only if it is bounded in absolute value.*

Proof. Note that for Archimedean places, $K_\mathfrak{p}$ is either $\mathbb{R}$ or $\mathbb{C}$, in which case it is just Heine-Borel's theorem. So, we assume that $K_\mathfrak{p}$ is non-Archimedean. For the $' \Rightarrow'$ direction, suppose $\overline{B}$ is compact. Owing to the fact that the absolute value map is a continuous map from $K_\mathfrak{p}$ to $\mathbb{R}$, the image of $\overline{B}$ under this map is also a compact set in $\mathbb{R}$ and hence the image is bounded.
For the other direction $' \Leftarrow'$, we suppose there is an integer d such that all elements of B have absolute value less than $(\mathbb{N}\mathfrak{p})^d$. This means every element of B is contained in $\varpi^{-d}\mathcal{O}_\mathfrak{p}$. Since $\mathcal{O}_\mathfrak{p}$ is compact and multiplication map is a continuous map, B is contained in a compact set. This in turn implies that B is relatively compact. $\square$

1.1. Additive characters and measures

The characters of the group $K_\mathfrak{p}^+$ can be explicitly expressed through the following

Proposition 2 ([Cas+76] Lemma 2.2.1). *Say $\xi \mapsto \chi(\xi)$ is a non-trivial character of $K_{\mathfrak{p}}^{+}$. Then, to each $\eta \in K_{\mathfrak{p}}^{+}$ we can associate an unitary character $\chi_\eta : \xi \mapsto \chi(\xi\eta)$. This correspondence is both a topological and algebraic isomorphism between $K_{\mathfrak{p}}^{+}$ and its character group $\widehat{K_{\mathfrak{p}}^{+}}$ (character group has compact open topology).*

Before we prove the proposition, we will remark a few things and also note certain results.

Remark 3. *Notice that the proposition requires the existence of a non-trivial character. We will first show that such a non-trivial character exists. Let p be the rational prime lying below $\mathfrak{p}$ and R be the completion of $\mathbb{Q}$ with respect to p. This leads us to two cases:*

1. *$\mathfrak{p}$ is Archimedean, in which case R is the real numbers. Define the map $\lambda : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ by $x \mapsto -x \mod 1$*

2. *$\mathfrak{p}$ is non-Archimedean, then $R = \mathbb{Q}_p$. Then, $\lambda : \mathbb{Q}_p \to \mathbb{R}/\mathbb{Z}$ is defined as follows: Let $x \in \mathbb{Q}_p$, then $x = p^{-v}(a_0 + a_1 p + \cdots)$ for an unique integer v. For $k \geq v$, we have*

$$\begin{aligned} p^k x &= p^{k-v}(a_0 + a_1 p + a_2 p^2 + \cdots) \\ &= \left(\frac{a_0 + a_1 p + a_2 p^2 + \cdots a_{v-1} p^{v-1}}{p^v} \right) p^k + p^k \mathbb{Z}_p \end{aligned}$$

   Choose an integer n such that $n \equiv p^k x \mod p^k \mathbb{Z}_p$. Then, put $\lambda(x) = n/p^k \in \left(\frac{\sum_{i=0}^{v-1} a_0 p^i}{p^v} \right) + \mathbb{Z}$. Hence, λ is determined uniquely upto mod 1.
   We will show that λ is determined by two properties:

   a) *$\lambda(x) \in \mathbb{Q}$ with only a p-power in denominator.*

   b) *$\lambda(x) - x \in \mathbb{Z}_p$.*

   Indeed, we can use property a to get $\lambda(x) = n/p^k$. Moreover, using property b we have $n - p^k x \in p^k \mathbb{Z}_p$ which implies $n \equiv p^k x \mod p^k \mathbb{Z}_p$. Hence, the properties are enough to find $\lambda(x)$ and by construction $\lambda(x)$ also has the two properties.

Lemma 4 ([Cas+76] Lemma 2.2.2). *The function $x \mapsto \lambda(x)$ is a non-trivial, continuous additive map of R into the group of reals mod 1.*

Proof. In the $\mathfrak{p}$ Archimedean case, this is obvious. So, let us focus on the $\mathfrak{p}$ non-Archimedean case. Note that $\lambda(x) + \lambda(y)$ is also a rational number with denominator p-power, so $\lambda(x) + \lambda(y)$ satisfies property a, and $\lambda(x) + \lambda(y) - (x + y) = (\lambda(x) - x) + (\lambda(y) - y) \in$

$\mathbb{Z}_p$ so $\lambda(x) + \lambda(y)$ satisfies property b. Hence, λ is additive. It is non-trivial since $\lambda(x) = 0 \Leftrightarrow x \in \mathbb{Z}_p$. To check continuity, it is enough to check at 0 since the map is additive. Let $\{x_n\}$ be a sequence going to 0 p-adically. Then, there exists $N \in \mathbb{Z}_{>0}$ such that $v_p(x_n) \geq 0 \ \forall \ n \geq N \Rightarrow x_n \in \mathbb{Z}_p$ whenever $n \geq N$. This implies $\lambda(x_n) = 0$ for $n \geq N$ from a previous observation. This completes the proof. □

Proof of 2. We have to check a few things:

1. *Well-defined*: For a fixed η, the map $\xi \mapsto \eta\xi$ is a continuous map and $\eta\xi \mapsto \chi(\eta\xi)$ is also a continuous map, so the map $\eta \mapsto \chi_\eta$ is a continuous and $|\chi(\eta\xi)| = 1$ which means the character χ_η is unitary. Also, $\chi_\eta(\xi_1 + \xi_2) = \chi(\eta(\xi_1 + \xi_2)) = \chi(\eta\xi_1 + \eta\xi_2) = \chi(\eta\xi_1)\chi(\eta\xi_2) = \chi_\eta(\xi_1)\chi_\eta(\xi_2)$, so χ_η is a group homomorphism.

2. The map $\eta \mapsto \chi_\eta$ is a group homomorphism. Indeed, $\chi_{\eta_1+\eta_2}(\xi) = \chi((\eta_1 + \eta_2)\xi) = \chi(\eta_1\xi + \eta_2\xi) = \chi(\eta_1\xi)\chi(\eta_2\xi) = \chi_{\eta_1}(\xi)\chi_{\eta_2}(\xi)$.

3. The map $\eta \mapsto \chi_\eta$ is injective. If η is in the kernel of the map, then $\chi_\eta(\xi) = 1 \ \forall \ \xi \in K_\mathfrak{p}^+ \Leftrightarrow \chi(\eta\xi) = 1 \ \forall \ \xi \in K_\mathfrak{p}^+ \Rightarrow \eta K_\mathfrak{p}^+ \neq K_\mathfrak{p}^+$ unless $\eta = 0$.

4. The map $\eta \mapsto \chi_\eta$ is bicontinuous. Since both the groups are topological groups, it is enough to check continuity at 0 and the trivial character respectively. Suppose $\{\eta_n\}$ is a sequence in $K_\mathfrak{p}^+$ going to 0. Let $W(C, U)$ with C compact and U open in S^1. Since χ is continuous, $\chi^{-1}(U)$ is open and hence there exists $\delta > 0$ such that $B(0;\delta) \subseteq \chi^{-1}(U)$. Since C is compact, therefore it is bounded in absolute value 1 by some integer M, i.e., $|x| < M$. Choose N such that $|\eta_n| < \delta/M$ for $n \geq N$. For $n \geq N$, we have $|\eta_n C| < \delta \Rightarrow \eta_n C \subseteq B(0;\delta) \subseteq \chi^{-1}(U) \Rightarrow \chi(\eta_n C) \subseteq U$ for all $n \geq N$ as required.
   Conversely, let $\{\chi_{\eta_n}\}$ be a sequence of characters tending to χ_{triv}. If $W(C, U)$ is a neighbourhood of χ_{triv}, then there exists N such that $\chi(\eta_n C) \subseteq U$ for all $n \geq N$. The above observation is independent of the choice of the compact set C and open set U. We will concern ourselves with $C_M = \{x \in K_\mathfrak{p}^+ : |x| \leq M\}$. Next, we choose U conveniently by first choosing $\xi \in K_\mathfrak{p}^+$, then choosing $U \ni 1$ such that $\chi(\xi) \notin U$. Then, from the observation made earlier in this paragraph, there exists N such that $\chi(\eta_n C_M) \subseteq U$ for all $n \geq N$. But $\chi(\xi) \notin U$, which means $\xi \notin \eta_n C_M$. Therefore, $|\xi| \geq |\eta_n| M \Rightarrow |\eta_n| \leq |\eta|/M$. But M was chosen arbitrarily, therefore η_n tends to 0 in $K_\mathfrak{p}^+$.

5. The map $\eta \mapsto \chi_\eta$ is surjective. Let H be the image of $K_\mathfrak{p}^+$ under the map. It is a locally compact subgroup of the Hausdorff space $\widehat{K_\mathfrak{p}^+} \Rightarrow H$ is a closed subgroup of $\widehat{K_\mathfrak{p}^+}$. We want to show $\widehat{K_\mathfrak{p}^+} = \overline{H}$ which is equivalent to showing that every character on $\widehat{K_\mathfrak{p}^+}/\overline{H}$ is the trivial character. Let ψ be such a character, then we can get a character of $\widehat{K_\mathfrak{p}^+}$ by the composition $\widehat{K_\mathfrak{p}^+} \to \widehat{K_\mathfrak{p}^+}/\overline{H} \to S^1$. By Pontryagin duality ($G \simeq \widehat{\widehat{G}}$ via $a \mapsto \delta_a : \chi \mapsto \chi(a)$) our character is also of this type. Moreover, our character vanishes on $\overline{H}$ which is equivalent to saying that it is trivial on all of the image of $\eta \mapsto \chi_\eta$ OR

$$\exists \xi \in K_\mathfrak{p}^+ \text{ such that } \chi_\eta(\xi) = \chi(\eta\xi) = 1 \; \forall \; \eta \in K_\mathfrak{p}^+$$

But multiplication is an automorphism of $K_\mathfrak{p}^+$, therefore $K_\mathfrak{p}^+\xi \neq K_\mathfrak{p}^+$ unless $\xi = 0$. Hence, ψ itself is trivial. Therefore, H is dense in $K_\mathfrak{p}^+$ and is closed as well. So, we conclude that $H = \widehat{K_\mathfrak{p}^+}$.

This completes the proof. □

Now, define for $\xi \in K_\mathfrak{p}^+$, $\Lambda(\xi) = \lambda(\mathrm{Tr}_{K_\mathfrak{p}/R}(\xi))$. We see that $\xi \mapsto \exp(2\pi i\Lambda(\xi))$ is a non-trivial character of $K_\mathfrak{p}^+$. Thus, we have proved

Theorem 5 ([Cas+76] Theorem 2.2.1)**.** *$K_\mathfrak{p}^+$ is naturally its own character group $\widehat{K_\mathfrak{p}^+}$ under the identification $\eta \mapsto (\xi \mapsto \exp(2\pi i\Lambda(\eta\xi)))$*

Lemma 6 ([Cas+76] Lemma 2.2.3)**.** *If $\mathfrak{p}$ is non-Archimedean, the character $\exp(2\pi i\Lambda(\eta\xi))$ associated to η is trivial on $\mathcal{O}_\mathfrak{p}$ if and only if $\eta \in \mathfrak{d}^{-1}$.*

Proof. The character $\exp(2\pi i\Lambda(\eta\xi))$ associated to η is trivial on $\mathcal{O}_\mathfrak{p}$ is equivalent to saying $\Lambda(\eta\mathcal{O}_\mathfrak{p}) = 0 \Leftrightarrow \lambda(\mathrm{Tr}_{K_\mathfrak{p}/R}(\eta\mathcal{O}_\mathfrak{p})) \Leftrightarrow \mathrm{Tr}_{K_\mathfrak{p}/R}(\eta\mathcal{O}_\mathfrak{p}) \subseteq \mathbb{Z}_p \Leftrightarrow \eta \in \mathfrak{d}^{-1}$. □

Since $K_\mathfrak{p}^+$ is a locally compact group, we have a Haar measure μ on it.

Lemma 7 ([Cas+76] Lemma 2.2.4)**.** *If we define $\mu_1(M) = \mu(\alpha M)$ for $0 \neq \alpha \in K_\mathfrak{p}$, and M a measurable set in $K_\mathfrak{p}$, then μ_1 is also a Haar measure, and consequently there exists a number $\mathrm{mod}\ \alpha > 0$ such that $\mu_1 = (\mathrm{mod}\ \alpha)\mu$.*

Proof. The multiplication map $\xi \mapsto \eta\xi$ is an automorphism of $K_\mathfrak{p}^+$ with $\xi \mapsto \alpha^{-1}\xi$ as the inverse. Since $K_\mathfrak{p}^+$ is a topological group, the multiplication map is bicontinuous as well. Therefore, it is both a topological and algebraic automorphism of $K_\mathfrak{p}^+$. If M is a measurable set and compact, then αM is compact as well and therefore $\mu(\alpha M) = \mu_1(M)$

is finite. If M is an open, measurable set, then αM is also open and measurable. By inner-regularity of μ on open sets, we have

$$\mu(\alpha M) = \sup\{\mu(K') : K' \subseteq \alpha M, K' \text{ compact}\}$$

Compact sets contained in αM are α-translates of the compact sets contained in M. This proves inner regularity of μ_1. Outer regularity also follows in a similar way. To see left invariance, observe that $\mu_1(M+\beta) = \mu(\alpha M + \alpha\beta) = \mu(\alpha M) = \mu_1(M)$. This completes the proof. □

Lemma 8 ([Cas+76] Lemma 2.2.5)**.** *The constant* $\mod \alpha$ *in the previous lemma is the absolute value* $|\alpha|$ *as defined in the beginning, i.e.,* $\mu(\alpha M) = |\alpha|\mu(M)$

Proof. It is clear that the definition of $\mod \alpha$ does not depend on the choice of the measurable set M. So, we can choose our set so as to ease calculations.

1. $K_\mathfrak{p} = \mathbb{R}$. Let $M = [0,1]$. Then, $\alpha M = [0,\alpha]$ for any $\alpha \in \mathbb{R}$. Here, the Haar measure is upto a scalar a Lebesgue measure, and the Lebesgue measure of $[0,\alpha]$ is $|\alpha|$. So, $\mu(\alpha M) = |\alpha|\mu(M)$
2. $K_\mathfrak{p} = \mathbb{C}$. Let $M = [0,1] \times [0,1]$. Then, αM is a square of area $|\alpha|^2$ and so $\mu(\alpha M) = |\alpha|\mu(M)$
3. K_{pr} is non-Archimedean. Let $M = \mathcal{O}_\mathfrak{p}$. Since M is open and compact, therefore $\mu(M) < \infty$. For $\alpha\mathcal{O}_\mathfrak{p}$, the number of cosets of $\alpha\mathcal{O}_\mathfrak{p}$ in $\mathcal{O}_\mathfrak{p}$ is given by $\mathbb{N}(\alpha\mathcal{O}_\mathfrak{p})$. Hence, $\mu(\alpha\mathcal{O}_\mathfrak{p})[\mathcal{O}_\mathfrak{p} : \alpha\mathcal{O}_\mathfrak{p}] = \mu(\mathcal{O}_\mathfrak{p})$.

This completes the proof. □

For the integral, we can interpret the previous lemma as follows:

$$d\mu(\alpha M) = |\alpha| d\mu(M)$$

Or,

$$\int_{K_\mathfrak{p}^+} f(\xi) d\mu(\xi) = |\alpha| \int_{K_\mathfrak{p}^+} f(\alpha\xi) d\mu(\xi)$$

This was for a general Haar measure. Now, we will try to choose our measure so that certain "nice" things happen. Let us see what we mean by "nice".

Let $d\mu$ be a Haar measure on $K_{\mathfrak{p}}^{+}$. Then, dual to this there is a measure $d\chi$ on $\widehat{K_{\mathfrak{p}}^{+}}$. Under the isomorphism $\widehat{\widehat{K_{\mathfrak{p}}^{+}}} \simeq K_{\mathfrak{p}}^{+}$, suppose $d\mu'$ is the measure corresponding to $d\chi$. We want to see how are $d\mu$ and $d\mu'$ related. Since $d\mu'$ is also a Haar measure, therefore there is a constant $c > 0$ such that $d\mu' = c\mu$. So, under the transition

$$K_{\mathfrak{p}}^{+} \longrightarrow \widehat{K_{\mathfrak{p}}^{+}} \longrightarrow K_{\mathfrak{p}}^{+}$$

$$d\mu \longrightarrow d\chi \longrightarrow d\mu' = c d\mu$$

$$\sqrt{c} d\mu \longrightarrow \frac{1}{\sqrt{c}} d\chi \longrightarrow \frac{c}{\sqrt{c}} d\mu$$

we want to choose our measure $d\mu$ such that under the transition $d\mu \mapsto d\chi \mapsto d\mu'$ the measure remains unchanged. We choose the following measures so that the above happens:

1. $d\xi =$ ordinary Lebesgue measure on the real line $\mathbb{R}$ if $K_{\mathfrak{p}}$ is real.
2. $d\xi =$ twice the ordinary Lebesgue measure if $K_{\mathfrak{p}}$ is complex.
3. $d\xi =$ the measure that gives $\mathcal{O}_{\mathfrak{p}}$ the value $(\mathbb{N}\mathfrak{d})^{-1}$ if $K_{\mathfrak{p}}$ is non-Archimedean.

All this discussion allows us to prove the following:

Theorem 9 ([Cas+76] Theorem 2.2.2)**.** *If we define the Fourier transform $\widehat{f}$ of a function $f \in L_1(K_{\mathfrak{p}}^{+})$ by :*

$$\widehat{f}(\eta) := \int_{K_{\mathfrak{p}}^{+}} f(\xi) \exp(-2\pi i \Lambda(\eta\xi)) d\xi$$

then with our choice of measure, we get the inversion formula

$$f(\xi) = \int_{K_{\mathfrak{p}}^{+}} \widehat{f}(\eta) \exp(2\pi i \Lambda(\eta\xi)) d\eta = \widehat{\widehat{f}}(-\xi)$$

Proof. We just need to check the formula for one non-trivial function, since any other function's transform will vary only upto a constant factor. The calculations can be seen in [Cas+76] §2.5. □

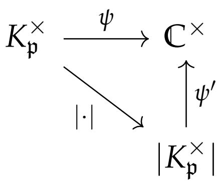
1.2. Multiplicative characters and measures

Now, let us investigate the characters on $K_\mathfrak{p}^\times$. Consider the continuous homomorphism $\alpha \mapsto |\alpha|$ from $K_\mathfrak{p}^\times$ to the group of positive real numbers. The kernel of this homomorphism is the set of all α such that $|\alpha| = 1$. We call the set U. It will play an important role in our analysis. U is compact and when $\mathfrak{p}$ is non-Archimedean, U is also open.

Definition 10. *We say a character χ of $K_\mathfrak{p}^\times$ is unramified if it is trivial on U. We will first find the unramified quasi-characters and use it to completely characterise the quasi-characters on $K_\mathfrak{p}^\times$.*

Lemma 11 ([Cas+76] Lemma 2.3.1)**.** *The unramified quasi-characters are the maps of the form $\chi(\alpha) = |\alpha|^s \equiv e^{s \log \alpha}$ where s is any complex number, s is determined by χ if $\mathfrak{p}$ is Archimedean and for $\mathfrak{p}$ non-Archimedean, s is defined upto mod $2\pi i / \log \mathbb{N}\mathfrak{p}$.*

Proof. Notice that for $\alpha \in U$, we have $|\alpha|^s = 1$, and $|\alpha\beta|^s = |\alpha|^s|\beta|^s$. The fact that $|\alpha|^s$ is a continuous map follows from absolute value map being a continuous homomorphism. Hence, $|\alpha|^s$ is indeed an unramified quasi-character. Let ψ be an unramified quasi-character, then such a quasi-character factors through a character ψ' of $|K_\mathfrak{p}^\times|$ ($|\alpha| \mapsto \psi(\alpha)$) as in the diagram below:



Clearly,

1. ψ' is a homomorphism. Indeed, $\psi'(|\alpha||\beta|) = \psi'(|\alpha\beta|) = \psi(\alpha\beta) = \psi(\alpha)\psi(\beta) = \psi'(|\alpha|)\psi'(|\beta|)$

2. ψ' is a character of the value group $|K_\mathfrak{p}^\times|$. Depending on whether $\mathfrak{p}$ is Archimedean or non-Archimedean, the value group $|K_\mathfrak{p}^\times|$ is $\mathbb{R}_{>0}$ or $(\mathbb{N}p)^\mathbb{Z}$.

Since the characters of $\mathbb{R}_{>0}$ is just raising to power s, we have $\psi(\alpha) = \psi'(|\alpha|) = |\alpha|^s; s \in \mathbb{C}$. If $s_1 \neq s_2 \in \mathbb{C}$, then raising to s_1 will give a different character than raising to s_2. So, every $s \in \mathbb{C}$ gives rise to a different character.
Now, to find the characters of $(\mathbb{N}\mathfrak{p})^\mathbb{Z}$, we only need to find character of $\mathbb{Z}$ as $(\mathbb{N}\mathfrak{p})^\mathbb{Z} \simeq \mathbb{Z}$ (the map being $(\mathbb{N}\mathfrak{p})^m \mapsto m$). A character of $\mathbb{Z}$ is $m \mapsto z^m$ with $z \in \mathbb{C}$. Write $z = re^{i\theta}$ in

the polar form with $r > 0$ and $0 \leq \theta < 2\pi$, put $x, y \in \mathbb{R}$ such that $\mathbb{N}\mathfrak{p}^x = r, \mathbb{N}\mathfrak{p}^y = e^\theta$. So, a character of $(\mathbb{N}\mathfrak{p})^{\mathbb{Z}}$ is $\mathbb{N}\mathfrak{p}^m \mapsto m \mapsto z^m = \mathbb{N}\mathfrak{p}^x \mathbb{N}\mathfrak{p}^{iy} = \mathbb{N}\mathfrak{p}^{x+iy} = \mathbb{N}\mathfrak{p}^s$ with $s = x + iy \in \mathbb{C}$. For $s_1 \neq s_2 \in \mathbb{C}$ that give rise to the same character, we have $\mathbb{N}\mathfrak{p}^{s_1} = \mathbb{N}\mathfrak{p}^{s_2} \Rightarrow \mathbb{N}\mathfrak{p}^{s_1 - s_2} = 1 \Rightarrow \exp((s_1 - s_2)\log \mathbb{N}\mathfrak{p}) = 1 \Rightarrow (s_1 - s_2)\log \mathbb{N}\mathfrak{p} = 2\pi i n; n \in \mathbb{Z} \Rightarrow s_1 - s_2 = (2\pi i / \log \mathbb{N}\mathfrak{p})n; n \in \mathbb{Z}$. This completes the proof. □

If $\mathfrak{p}$ is Archimedean, we can write a general element $\alpha \in K_{\mathfrak{p}}^\times$ uniquely in the form $\alpha = \tilde{\alpha}\rho, \tilde{\alpha} \in U, \rho > 0$. Indeed let $\rho = |\alpha|, \tilde{\alpha} = \alpha/|\alpha|$. If $\mathfrak{p}$ is non-Archimedean, then $\alpha = \tilde{\alpha}\rho$ with $\tilde{\alpha} \in U, \rho$ a power of ϖ. Hence, the map $\alpha \mapsto \tilde{\alpha}$ is a continuous homomorphism from $K_{\mathfrak{p}}^\times$ to U.

Theorem 12 ([Cas+76] Theorem 2.3.1). *The quasi-characters of $K_{\mathfrak{p}}^\times$ are the maps of the form $\alpha \mapsto \chi(\alpha) = \tilde{\chi}(\tilde{\alpha})|\alpha|^s$, where $\tilde{\chi}$ is a character of U. $\tilde{\chi}$ is uniquely determined by χ and s is determined as in previous lemma.*

Proof. A map of the given type is definitely a quasi-character. Conversely, if χ is a quasi-character, then we define $\tilde{\chi}$ to be the restriction of χ to U and is therefore a character of U since U is compact. The map $\alpha \mapsto \chi(\alpha)/\tilde{\chi}(\tilde{\alpha})$ is an unramified quasi-character, therefore it is of the form $|\alpha|^s$ with s as determined in previous lemma. □

Remark 13. *Two quasi-characters are said to be equivalent if their quotient is an unramified quasi-character.*

So, the search for quasi-characters of $K_{\mathfrak{p}}^\times$ can be completed if we figure out the characters of U.

1. For $\mathfrak{p}$ Archimedean and real, we have $U = \{\pm 1\}$ and the character is $\tilde{\chi}$ such that $\tilde{\chi}(-1)^2 = \tilde{1} = 1 \Rightarrow \tilde{\chi}(-1)$ is a square root of unity. Hence, $\tilde{\chi}(\tilde{\alpha}) = \tilde{\alpha}^n; n = 0, 1$.

2. If $\mathfrak{p}$ is Archimedean and complex, then $U = \{z \in \mathbb{C} : |z| = 1\}$. And, the characters of this group are $\tilde{\chi}(\tilde{\alpha}) = \tilde{\alpha}^n; n \in \mathbb{Z}$.

3. When $\mathfrak{p}$ is non-Archimedean, we know that $1 + \mathfrak{p}^n; n > 0$ forms a fundamental system of neighbourhoods of 1. Now, take a neighbourhood V of $1 \in \mathbb{C}^\times$ such that it only contains the trivial subgroup (no small subgroups theorem), then $\tilde{\chi}^{-1}(V)$ is an open neighbourhood of $1 \in K_{\mathfrak{p}}^\times$ and so there exists $v > 0$ such that $1 + \mathfrak{p}^v \subseteq \tilde{\chi}^{-1}(V) \Rightarrow \chi(1 + \mathfrak{p}^v) \subseteq U$ is a subgroup but by choice there is only the trivial subgroup, so $\tilde{\chi}(1 + \mathfrak{p}^v) = 1$ for sufficiently large v. The minimal such v allows us to define the conductor $\mathfrak{f} = \mathfrak{p}^v$ of $\tilde{\chi}$. Then, $\tilde{\chi}$ may be described as a character of the finite factor group $U/1 + \mathfrak{f}$.

Now, let us move on to the measure on $K_\mathfrak{p}^\times$. We will use the measure $d\xi$ on $K_\mathfrak{p}^+$ to construct a measure $d\alpha$ on $K_\mathfrak{p}^\times$. If $f(\alpha) \in L(K_\mathfrak{p}^\times)$, then $f(\xi)|\xi|^{-1} \in L(K_\mathfrak{p}^+ - \{0\})$. So, we may defined a functional

$$\Phi(f) := \int_{K_\mathfrak{p}^+ - \{0\}} f(\xi)|\xi|^{-1} d\xi$$

If $g(\alpha) = f(\beta\alpha)$ for some fixed $\beta \in K_\mathfrak{p}^\times$, then

$$\Phi(g) = \int_{K_\mathfrak{p}^+ - \{0\}} f(\beta\xi)|\xi|^{-1} d\xi = \Phi(f)$$

after appropriate substitutions. Therefore, our functional is invariant under multiplicative transform and hence must come from a Haar measure on $K_\mathfrak{p}^\times$. Denote this measure by $d^\times\alpha$. We obtain

$$\int_{K_\mathfrak{p}^\times} f(\alpha) d^\times\alpha = \int_{K_\mathfrak{p}^+ - \{0\}} f(\xi)|\xi|^{-1} d\xi$$

The $1-1$ correspondence between $L(K_\mathfrak{p}^+ - \{0\})$ and $L(K_\mathfrak{p}^\times)$, and viewing the functions of $L_1(K_\mathfrak{p}^+ - \{0\})$ and $L_1(K_\mathfrak{p}^\times)$ as limits of the functions in the previous spaces (respectively), we have

Lemma 14 ([Cas+76] Lemma 2.3.2)**.** *$f(\alpha) \in L_1(K_\mathfrak{p}^\times) \Leftrightarrow f(\xi)|\xi|^{-1} \in L_1(K_\mathfrak{p}^+ - \{0\})$, and for these functions, we have*

$$\int_{K_\mathfrak{p}^\times} f(\alpha) d^\times\alpha = \int_{K_\mathfrak{p}^+ - \{0\}} f(\xi)|\xi|^{-1} d\xi$$

We normalise the measure $d^\times\alpha$ such that it gives measure 1 on U. This is done by the following choice:

1. If $\mathfrak{p}$ is Archimedean, then $d^\times\alpha = |\alpha|^{-1} d\alpha$
2. If $\mathfrak{p}$ is non-Archimedean, then $d^\times\alpha = \dfrac{\mathbb{N}\mathfrak{p}}{\mathbb{N}\mathfrak{p} - 1} \dfrac{d\alpha}{|\alpha|}$

1.3. The local ζ function and functional equation

Let us define an important class of functions. By $\mathfrak{Z}$, we denote the class of functions satisfying the following properties

1. $f(\xi)$ and $\widehat{f}(\xi)$ are continuous, and belong to $L_1(K_\mathfrak{p}^+)$

2. $f(\alpha)|\alpha|^{\sigma}$ and $\widehat{f}(\alpha)|\alpha|^{\sigma}$ belong to $L_1(K_{\mathfrak{p}}^{+})$ for $\sigma = \mathrm{Re}(s) > 0$

Definition 15. *For $f \in \mathfrak{Z}$ and quasicharacters χ with exponent > 0, we introduce a function $\zeta(f,\chi)$ as*

$$\zeta(f,\chi) = \int_{K_{\mathfrak{p}}^{\times}} f(\alpha)\chi(\alpha)d^{\times}\alpha$$

and call such a function a ζ-function of $K_{\mathfrak{p}}$.

Lemma 16 ([Cas+76] Lemma 2.4.1)**.** *A ζ-function is regular for all quasi-characters with of exponent greater than 0.*

Proof. First consider the case when F is Archimedean. In this case we note that

$$\int_{K_{\mathfrak{p}}^{\times}} |f(\alpha)\chi(\alpha)|d^{\times}\alpha = \int_{K_{\mathfrak{p}}^{\times}} |f(\alpha)||\alpha|^{\sigma}d^{\times}\alpha$$

But this integral is finite by the assumption that $f \in \mathfrak{Z}$.
For the non-Archimedean case, observe that f is locally constant with compact support and thus factors through a finite quotient group of the form $\varpi^m\mathcal{O}_{\mathfrak{p}}/\varpi^n\mathcal{O}_{\mathfrak{p}}$ for some integers m,n. By the linearity and translational invariance of Haar measure, it suffices to check only for functions f that are characteristic functions of $\varpi^k\mathcal{O}_{\mathfrak{p}}$. Note that

$$\varpi^k \setminus \{0\} = \bigsqcup_{j \geq k} \varpi^j\mathcal{O}_{\mathfrak{p}}^{\times}$$

and therefore

$$\begin{aligned} I &= \int_{K_{\mathfrak{p}}^{\times}} |f(\alpha)||\alpha|^{\sigma}d^{\times}\alpha \\ &= \mathrm{Meas}_{d^{\times}\alpha}(\mathcal{O}_{\mathfrak{p}}^{\times}) \sum_{j\geq k} q^{-j\sigma} \\ &= \mathrm{Meas}_{d^{\times}\alpha}(\mathcal{O}_{\mathfrak{p}}^{\times}) \frac{q^{-k\sigma}}{1-q^{-\sigma}} < \infty \end{aligned}$$

This completes the proof. □

Lemma 17 ([Cas+76] Lemma 2.4.2)**.** *For χ in the domain $0 < \sigma < 1$ and $\chi^{\vee} = |\alpha|\chi^{-1}(\alpha)$, we have*

$$\zeta(f,\chi)\zeta(\widehat{g},\chi^{\vee}) = \zeta(\widehat{f},\chi^{\vee})\zeta(g,\chi)$$

for all $f, g \in \mathfrak{Z}$

Proof.

$$\zeta(f,\chi)\zeta(\hat{g},\chi^\vee) = \int_{K_\mathfrak{p}^\times} f(\alpha)\chi(\alpha)d^\times\alpha \int_{K_\mathfrak{p}^\times} \hat{g}(\beta)\chi^{-1}(\beta)|\beta|d^\times\beta$$

If the exponent of χ is σ, then the exponent of $\chi^\vee(\alpha)$ is $1-\sigma$. Hence, both σ and $1-\sigma$ are both in the region of definition as in the hypothesis. Moreover, due to absolute convergence we can club the integrals to get

$$= \int_{K_\mathfrak{p}^\times}\int_{K_\mathfrak{p}^\times} f(\alpha)\chi(\alpha)\hat{g}(\beta)\chi^{-1}(\beta)|\beta|d^\times\alpha d^\times\beta$$

Since $d^\times\alpha d^\times\beta$ is invariant under multiplicative translation, therefore the map $(\alpha,\beta)\mapsto(\alpha,\alpha\beta)$ is an automorphism. We get

$$\begin{aligned} &= \int_{K_\mathfrak{p}^\times}\int_{K_\mathfrak{p}^\times} f(\alpha)\chi(\alpha)\hat{g}(\alpha\beta)\chi^{-1}(\alpha\beta)|\alpha\beta|d^\times\alpha d^\times\beta \\ &= \int_{K_\mathfrak{p}^\times}\left(\int_{K_\mathfrak{p}^\times} f(\alpha)\hat{g}(\alpha\beta)|\alpha|d^\times\alpha\right)\chi^{-1}(\beta)|\beta|d^\times\beta \end{aligned}$$

Now, the term outside the inner integral is independent of f,g. So, it is enough to show that the inner integral is symmetric in f,g.

$$\begin{aligned} \int_{K_\mathfrak{p}^\times} f(\alpha)\hat{g}(\alpha\beta)|\alpha|d^\times\alpha &= \int_{K_\mathfrak{p}^+} f(\xi)\hat{g}(\xi\beta)d\xi \\ &= \int_{K_\mathfrak{p}^+} f(\xi)\int_{K_\mathfrak{p}^+} g(\eta)\exp(-2\pi i\Lambda(\eta\xi\beta))d\xi d\eta \\ &= \int_{K_\mathfrak{p}^+}\int_{K_\mathfrak{p}^+} f(\xi)g(\eta)\exp(-2\pi i\Lambda(\eta\xi\beta))d\xi d\eta \end{aligned}$$

And the above is symmetric in f,g. □

Theorem 18 (Main theorem of local theory). *[[Cas+76] Theorem 2.4.1] A ζ-function has an analytic continuation to the domain of all quasi-characters given by the functional equation of the type*

$$\zeta(f,\chi) = \rho(\chi)\zeta(\hat{f},\chi^\vee)$$

The factor $\rho(\chi)$ is independent of the choice of f, is a meromorphic function of quasi-characters defined for $0<\sigma<1$ by the functional equation itself and for all quasi-characters by analytic continuation.

Proof. Fix an equivalence class of characters C, and choose a function f_C. Then, for any

$f \in \mathfrak{Z}$ we have

$$\zeta(f,\chi)\zeta(\widehat{f_C},\chi^\vee) = \zeta(\widehat{f},\chi^\vee)\zeta(f_C,\chi)$$
$$\frac{\zeta(f,\chi)}{\widehat{f},\chi^\vee} = \frac{\zeta(\widehat{f_C},\chi^\vee)}{\zeta(f_C,\chi)}$$

Defining $\rho(\chi) := \dfrac{\zeta(\widehat{f_C},\chi^\vee)}{\zeta(f_C,\chi)}$, we note that this $\rho(\chi)$ is defined on $0 < \sigma < 1$ by previous lemma. An explicit computation of $\rho(\chi)$ (in next section) will tell us that it is actually a meromorphic function in the parameter s defined over the class C and hence we will have an analytic continuation to all of C. The theorem then follows from this. □

1.4. Computation of $\rho(\chi)$ for special functions

Before computing the value of $\rho(\chi)$ for special functions. Let us note a few things (corollaries to the main theorem 18) about it.

1. $\rho(\chi^\vee) = \chi(-1)/\rho(\chi)$. Indeed, $\zeta(f,\chi) = \rho(\chi)\zeta(\widehat{f},\chi^\vee) = \rho(\chi)\rho(\chi^\vee)\zeta(\widehat{\widehat{f}},\chi^{\vee\vee}) = \rho(\chi)\rho(\chi^\vee)\chi(-1)\zeta(f,\chi)$ since $\widehat{\widehat{f}}(\alpha) = f(-\alpha)$ and $\chi^{\vee\vee}(\alpha) = c(\alpha)$. This implies $\rho(\chi)\rho(\chi^\vee) = \chi(-1)$.

2. $\chi(\overline{\chi}) = \chi(-1)\overline{\rho(\chi)}$. Indeed, note that $\widehat{\overline{f}}(\alpha) = \overline{\widehat{f}}(\alpha)$ and $\widehat{\chi}^\vee(\alpha) = \overline{\chi}^\vee(\alpha)$. Therefore, $\overline{\zeta(f,c)} = \zeta(\overline{f},\overline{\chi}) = \rho(\overline{\chi})\zeta(\widehat{\overline{f}},\overline{\chi}^\vee) = \rho(\overline{\chi})\chi(-1)\zeta(\overline{\widehat{f}},\overline{\chi^\vee}) = \rho(\overline{\chi})\rho(-1)\overline{\zeta(\widehat{f},\chi^\vee)}$. But, $\overline{\zeta(f,\chi)} = \overline{\rho(\chi)}\overline{\zeta(\widehat{f},\chi^\vee)}$. This allows us to conclude that $\rho(\overline{\chi}) = \chi(-1)\overline{\rho(\chi)}$.

3. $|\rho(\chi)| = 1$ for χ of exponent $1/2$. To see this, note that exponent of $\chi = 1/2$ implies $\chi(\alpha)\overline{\chi}(\alpha) = |\chi(\alpha)|^2 = |\alpha| = \chi(\alpha)\chi^\vee(\alpha) \Rightarrow \overline{\chi}(\alpha) = \chi^\vee(\alpha)$. After this, use the last two results.

All the calculations are done in Tate's original thesis [Cas+76] §2.5, so I will not repeat the arguments. I will just tabulate the data. I will do that in a case by case manner, first $K_\mathfrak{p}$ real, then complex and then $\mathfrak{p}$-adic.

$K_\mathfrak{p}$ is real

The equivalence classes of quasi-characters are $|\alpha|^s$ and $\pm|\alpha|^s$. The two classes will be represented by $||^s$ and $\pm||^s$ respectively. Note that $\rho(\chi)$ as in the main theorem, does not depend on the choice of f_C so we choose our f_C conveniently. The corresponding functions are

- $f_{||^s}(\xi) = \exp(-\pi\xi^2)$
- $f_{\pm||^s}(\xi) = \xi \exp(-\pi\xi^2)$

Their Fourier transforms are

- $\widehat{f_{||^s}}(\xi) = f(\xi)$
- $\widehat{f_{\pm||^s}}(\xi) = i f_{\pm||^s}(\xi)$

The corresponding ζ-functions are:

- $\zeta(f_{||^s}, ||^s) = \pi^{-s/2}\Gamma(s/2)$
- $\zeta(f_{\pm||^s}, \pm||^s) = \pi^{-\frac{s+1}{2}}\Gamma((s+1)/2)$
- $\zeta(\widehat{f_{||^s}}, (||_s)^\vee) = \pi^{-\frac{1-s}{2}}\Gamma((1-s)/2)$
- $\zeta(\widehat{f_{\pm||^s}}, (\pm||^s)^\vee) = i\pi^{-\frac{(1-s)+1}{2}}\Gamma(((1-s)+1)/2)$

This gives the explicit expression for $\rho(\chi)$:

- $\rho(||^s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)$
- $\rho(\pm||^s) = -i2^{1-s}\pi^{-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(s)$

$K_\mathfrak{p}$ is complex

The equivalence classes of quasi-characters are $\chi_n||^s$ where $\chi_n(\alpha) = \chi_n(re^{i\theta}) = e^{in\theta}$. The corresponding functions are

- $f_n(\xi = x+iy) = (x-iy)^{|n|}e^{-2\pi(x^2+y^2)}$ if $n \geq 0$
- $f_n(\xi = x+iy) = (x+iy)^{|n|}e^{-2\pi(x^2+y^2)}$ if $n \leq 0$

Their Fourier transform is

$$\widehat{f_n}(\xi) = i^{|n|} f_{-n}(\xi) \text{ for all } n$$

The ζ-functions are: For $\alpha = re^{i\theta}$

- $\zeta(f_n, \chi_n||^s) = (2\pi)^{1-s+\frac{|n|}{2}}\Gamma\left(s + \frac{|n|}{2}\right)$
- $\zeta(\widehat{f_n}, (\chi_n||^s)^\vee) = i^{|n|}(2\pi)^{s+\frac{|n|}{2}}\Gamma\left(1 - s\frac{|n|}{2}\right)$

Therefore, the explicit expression for $\rho(\chi)$ is

$$\rho(\chi) = (-i)^{|n|} \frac{(2\pi)^{1-s}\Gamma\left(s + \frac{|n|}{2}\right)}{(2\pi)^s\Gamma\left(1 - s + \frac{|n|}{2}\right)}$$

$K_{\mathfrak{p}}$ is $\mathfrak{p}$-adic

The equivalence classes are represented by χ_n for $n \geq 0$. It denotes a character of $K_{\mathfrak{p}}^{\times}$ with conductor exactly $\mathfrak{p}^n$ such that $\chi_n(\varpi) = 1$. The corresponding functions are

- $f_n(\xi) = \exp(2\pi i \Lambda(\xi))$ for $\xi \in \mathfrak{d}^{-1}\mathfrak{p}^{-n}$
- $f_n(\xi) = 0$ otherwise

Their Fourier transforms are

- $\widehat{f}_n(\xi) = (\mathbb{N}\mathfrak{d})^{1/2}(\mathbb{N}\mathfrak{p})^n$ for $\xi \equiv 1 \pmod{\mathfrak{p}}^n$
- $\widehat{f}_n(\xi) = 0$ otherwise

The ζ-functions are

- For $n = 0$, we have $\zeta(f_0, ||^s) = \dfrac{\mathbb{N}\mathfrak{d}^{s-1/2}}{1 - \mathbb{N}\mathfrak{p}^{-s}}$ and $\zeta(\widehat{f}_0, (||^s)^{\vee}) = \dfrac{1}{1 - \mathbb{N}\mathfrak{p}^{s-1}}$
- For $n > 0$, if $\{\epsilon\}$ are representatives of the factor group $U/1 + \mathfrak{p}^n$. Then, we have

$$\zeta(f_n) = \mathbb{N}\mathfrak{p}^{(d+n)^s} \left(\sum_{\epsilon} \chi_n(\epsilon) \exp(2\pi i \Lambda(\epsilon/\pi^{d+n}))\right) \int_{1+\mathfrak{p}^n} d^{\times}\alpha$$

  and

$$\zeta(\widehat{f}_n, (\chi_n||^s)^{\vee}) = \mathbb{N}\mathfrak{d}^{1/2}\mathbb{N}\mathfrak{p}^n \int_{1+\mathfrak{p}^n} d^{\times}\alpha$$

This allows us to get an explicit form of $\rho(\chi)$ which is

$$\rho(||^s) = \mathbb{N}\mathfrak{d}^{s-1/2} \frac{1 - \mathbb{N}\mathfrak{p}^{s-1}}{1 - \mathbb{N}\mathfrak{p}^{-s}}$$

and

$$\rho(\chi_n||^s) = \mathbb{N}(\mathfrak{d}\mathfrak{f})^{s-1/2}\rho_0(\chi)$$

if χ is a ramified character of conductor $\mathfrak{f}$ such that $\chi(\varpi) = 1$, and $\rho_0(\chi) = \mathbb{N}\mathfrak{f}^{-1/2} \sum_{\epsilon} c(\epsilon) \exp(2\pi i \Lambda(\epsilon$

Remark 19. *The $\rho_0(\chi)$ above is the so-called root number and has absolute value* 1.

2. Global Theory

2.1. Characters and measures

2.1.1. Characters

We wish to view every quasi-character c of $\mathbb{A}_K$ in terms of quasi-characters of the local fields K_v. If $\chi(\mathfrak{a})$ is a quasi-character of G, i.e. a continuous multiplicative map from G to $\mathbb{C}^\times$, we denote by $\chi_\mathfrak{p}$ the restriction of χ to $G_\mathfrak{p}$: $G_\mathfrak{p} \ni \mathfrak{a}_\mathfrak{p} \mapsto \chi(\mathfrak{a}_\mathfrak{p}) = \chi(\cdots 1, \mathfrak{a}_\mathfrak{p}, 1, \cdots)$. Clearly, $\chi_\mathfrak{p}$ is a quasi-character of $G_\mathfrak{p}$. In fact,

Lemma 20 ([Cas+76] Lemma 3.2.1)**.** *$\chi_\mathfrak{p}$ is trivial on $H_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$, and we have for any $\mathfrak{a} \in G$*

$$\chi(\mathfrak{a}) = \prod_\mathfrak{p} \chi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$$

with all but finitely many factors in the product being 1

Proof. Let V be a neighbourhood of 1 in $\mathbb{C}^\times$ containing no multiplicative subgroup other than the trivial one. Then, $U = \prod U_\mathfrak{p}$ is a neighbourhood of $1 \in G$ such that $\chi(U) \subseteq V$. Let S be the finite set of $\mathfrak{p}$ for which $U_\mathfrak{p} \neq H_\mathfrak{p}$, then $G^S \subseteq N \Rightarrow \chi(G^S) \subseteq \chi(N) \subseteq V \Rightarrow \chi(G^S) = 1 \Rightarrow \chi(H_\mathfrak{p}) = 1$ for all $\mathfrak{p} \notin S$. If $\mathfrak{a}$ is a fixed element of G, then we can write $\mathfrak{a} = \prod_{\mathfrak{p} \in S} \mathfrak{a}_\mathfrak{p} \times \mathfrak{a}^S$ such that $\mathfrak{a}^S \in G^S$. Then,

$$\chi(\mathfrak{a}) = \prod_{\mathfrak{p} \in S} \chi(\mathfrak{a}_\mathfrak{p}) \times \chi(\mathfrak{a}^S) = \prod_{\mathfrak{p} \in S} \chi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}) = \prod_\mathfrak{p} \chi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$$

as for $\mathfrak{p} \notin S$ we have $\chi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}) = 1$. □

Moreover, the converse is also true.

Lemma 21 ([Cas+76] Lemma 3.2.2)**.** *Let $\chi_\mathfrak{p}$ be a quasi-character of $G_\mathfrak{p}$ for all $\mathfrak{p}$ with $\chi_\mathfrak{p}$ trivial on $H_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$. Then, if we define $\chi(\mathfrak{a}) = \prod_\mathfrak{p} \chi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ we obtain a quasi-character of G.*

Proof. Let S be the finite set of places such that $\chi_\mathfrak{p}$ is trivial on $H_\mathfrak{p}$. If s is the cardinality of S, then for a neighbourhood U of 1 in $\mathbb{C}^\times$ there exists neighbourhood V such that $V^s \subseteq U$. Let $N_\mathfrak{p}$ be the neighbourhood of 1 in $G_\mathfrak{p}$ such that $\chi_\mathfrak{p}(N_\mathfrak{p}) \subseteq V$ for $\mathfrak{p} \in S$ and let $N_\mathfrak{p} = H_\mathfrak{p}$ for $\mathfrak{p} \notin S$. Then,

$$\chi\left(\prod_\mathfrak{p} N_\mathfrak{p}\right) \subseteq V^s \subseteq U$$

Hence, χ is continuous. This completes the proof. □

Notice that $\chi(\mathfrak{a}) = \prod_\mathfrak{p} \chi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ is a character if and only if $\chi_\mathfrak{p}$ is a character. Let $\widehat{G_\mathfrak{p}}$ be the set of characters of $G_\mathfrak{p}$ for all $\mathfrak{p}$, and let $H_\mathfrak{p}^*$ be the subgroup of $\widehat{G_\mathfrak{p}}$ consisting of elements $\chi_\mathfrak{p}$ which are trivial on $H_\mathfrak{p}$. Then, $H_\mathfrak{p}$ is compact and implies that $\widehat{H_\mathfrak{p}} \simeq \widehat{G_\mathfrak{p}}/H_\mathfrak{p}^*$ is discrete $\Rightarrow$ $H_\mathfrak{p}^*$ is open, and $H_\mathfrak{p}$ open implies $G_\mathfrak{p}/H_\mathfrak{p}$ is discrete and this means $\widehat{G_\mathfrak{p}/H_\mathfrak{p}} \simeq H_\mathfrak{p}^*$ is compact.

Theorem 22 ([Cas+76] Theorem 3.2.1)**.** *The restricted product topology of $\widehat{G_\mathfrak{p}}$ with respect to the subgroups $H_\mathfrak{p}^*$ is naturally isomorphic to the character group $\widehat{G}$ of G, and this isomorphism is both algebraic and topological.*

Proof. The above two lemmas show that the isomorphism is algebraic. Now, we want to show topological isomorphism. For this, note that $\chi = (\cdots, \chi_\mathfrak{p}, \cdots)$ is close to the identity character $\Leftrightarrow$ $\chi(F)$ is close to 1 for a large enough compact set F $\Leftrightarrow$ $\chi(\prod_\mathfrak{p} F_\mathfrak{p})$ is close to 1 for $F_\mathfrak{p} \subseteq G_\mathfrak{p}$ compact, and $F_\mathfrak{p} = H_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$ $\Leftrightarrow$ $\chi_\mathfrak{p}(F_\mathfrak{p})$ is close to 1 whenever $F_\mathfrak{p} \neq H_\mathfrak{p}$ and $\chi_\mathfrak{p}(F_\mathfrak{p}) = \chi_\mathfrak{p}(H_\mathfrak{p}) = 1$ for the remaining $\mathfrak{p}$ $\Leftrightarrow$ $\chi_\mathfrak{p}$ is close to 1 in $\widehat{G_\mathfrak{p}}$ for a finite number of $\mathfrak{p}$ and $\chi_\mathfrak{p} \in H_\mathfrak{p}^*$ at other $\mathfrak{p}$ $\Leftrightarrow$ χ is close to 1 in the restricted direct product topology of $\widehat{G_\mathfrak{p}}$. □

2.1.2. Measures

For each $G_\mathfrak{p}$, choose a measure $d\mathfrak{a}_\mathfrak{p}$ such that $\int_{H_\mathfrak{p}} d\mathfrak{a}_\mathfrak{p} = 1$ for all but finitely many $\mathfrak{p}$. We want to define measure $d\mathfrak{a}$ on G for we can have $d\mathfrak{a} = \prod_\mathfrak{p} d\mathfrak{a}_\mathfrak{p}$. First, choose a finite set S, then $G_S = \left(\prod_{\mathfrak{p}\in S} G_\mathfrak{p}\right) \times G^S$. Let $d\mathfrak{a}^S$ be the measure on G^S such that $\int_{G^S} d\mathfrak{a}^S = \prod_{\mathfrak{p}\notin S} \int_{H_\mathfrak{p}} d\mathfrak{a}_\mathfrak{p}$. This allows us to define a measure $d\mathfrak{a}_S$ as $\prod_\mathfrak{p} d\mathfrak{a}_\mathfrak{p} d\mathfrak{a}^S$. Since, G_S is an open subset of G, the measure on G is determined by its value on G_S. The measure $d\mathfrak{a}$ on G is therefore defined such that $d\mathfrak{a} = d\mathfrak{a}_S$. A priori it looks like our measure $d\mathfrak{a}$ depends on the choice of S. We will show that this is not true. Let T be another set containing

S, then $G_S \subseteq G_T$ and we only need to check that $d\mathfrak{a}_T$ coincides with $d\mathfrak{a}_S$. This can be verified easily as

$$d\mathfrak{a}_S = \prod_{\mathfrak{p}\in S} d\mathfrak{a}_\mathfrak{p} d\mathfrak{a}^S = \prod_{\mathfrak{p}\in S} d\mathfrak{a}_\mathfrak{p} \prod_{\mathfrak{p}\in T\setminus S} d\mathfrak{a}_\mathfrak{p} d\mathfrak{a}^T = d\mathfrak{a}_T$$

We have therefore constructed a measure $d\mathfrak{a}$ on G which is going to be denoted by $d\mathfrak{a}$.

Lemma 23 ([Cas+76] Lemma 3.3.2). *Suppose for each $\mathfrak{p}$, we have a continuous function $f_\mathfrak{p} \in L_1(G_\mathfrak{p})$ such that $f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}) = 1$ on $H_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$. Then the function $f(\mathfrak{a}) = \prod_\mathfrak{p} f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ is:*

1. *$f(\alpha)$ is continuous on G*

2. *For any set S containing $\mathfrak{p}$ such that $f_\mathfrak{p}(H_\mathfrak{p}) \neq 1$ or $\int_{H_\mathfrak{p}} d\mathfrak{a}_\mathfrak{p} \neq 1$, we have*

$$\int_{G_S} f(\mathfrak{a}) d\mathfrak{a} = \prod_{\mathfrak{p}\in S} \left(\int_{G_\mathfrak{p}} f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}) d\mathfrak{a}_\mathfrak{p} \right)$$

Proof. 1. $f(\mathfrak{a})$ is continuous on G_S and therefore it is continuous on G as well.

2. For $\mathfrak{a} \in G_S$, we have $f(\mathfrak{a}) = \prod_\mathfrak{p} f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$. Therefore,

$$\begin{aligned} \int_{G_S} f(\mathfrak{a}) d\mathfrak{a} &= \int_{G_S} f(\mathfrak{a}) d\mathfrak{a}_S = \int_{G_S} \prod_\mathfrak{p} f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}) \prod_{\mathfrak{p}\in S} d\mathfrak{a}^S \\ &= \prod_{\mathfrak{p}\in S} \int_{G_\mathfrak{p}} f(\mathfrak{a}_\mathfrak{p}) d\mathfrak{a}_\mathfrak{p} \int_{G^S} d\mathfrak{a}^S \\ &= \prod_{\mathfrak{p}\in S} \int_{G_\mathfrak{p}} f(\mathfrak{a}_\mathfrak{p}) d\mathfrak{a}_\mathfrak{p} \end{aligned}$$

□

Theorem 24 ([Cas+76] Theorem 3.3.1). *If $f(\mathfrak{a})$ and $f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ is as in the previous lemma, and if further*

$$\prod_\mathfrak{p} \int_{G_\mathfrak{p}} |f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})| d\mathfrak{a}_\mathfrak{p} < \infty$$

then $f(\mathfrak{a}) \in L_1(G)$ and

$$\int_G f(\mathfrak{a}) d\mathfrak{a} = \prod_\mathfrak{p} \int_{G_\mathfrak{p}} f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}) d\mathfrak{a}_\mathfrak{p}$$

Lemma 25 ([Cas+76] Lemma 3.3.3)**.** *If $f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ are measurable functions in $G_\mathfrak{p}$ for all $\mathfrak{p}$ and $f_{\mathfrak{a}_\mathfrak{p}}$ is the characteristic function of $H_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$, then the function $\prod_\mathfrak{p} f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ has the Fourier transform $\widehat{f}(\chi) = \prod_\mathfrak{p} \widehat{f}_\mathfrak{p}(\chi_\mathfrak{p})$ and $f(\mathfrak{a})$ is also measurable.*

This lemma requires us to talk of the dual measure on the character group. So, let us move ahead with that. Denote by $\chi = (\cdots, \chi_\mathfrak{p}, \cdots)$ an element of $\widehat{G}$ (these are characters not quasi-characters). Let $d\chi_\mathfrak{p}$ be the measure in $\widehat{G_\mathfrak{p}}$ dual to $d\mathfrak{a}_\mathfrak{p}$ in $G_\mathfrak{p}$. Since $f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ is the characteristic function on $H_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$, therefore the Fourier transform of such a $f_\mathfrak{p}$ is $\widehat{f}_\mathfrak{p}(\chi_\mathfrak{p}) = \int_{G_\mathfrak{p}} f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})\overline{\chi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})}d\mathfrak{a}_\mathfrak{p} = \int_{H_\mathfrak{p}} \overline{\chi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})}d\mathfrak{a}_\mathfrak{p}$. This integral amounts to 0 if $\chi_\mathfrak{p}$ is non-trivial (by orthogonality properties) and it is equal to $\int_{H_\mathfrak{p}} d\mathfrak{a}_\mathfrak{p}$ if $\chi_\mathfrak{p}$ is trivial on $H_\mathfrak{p}$. Applying Fourier transform gives us

$$\begin{aligned} f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p}) &= \int_{\widehat{G_\mathfrak{p}}} \widehat{f}_\mathfrak{p}(\chi_\mathfrak{p})\chi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})d\chi_\mathfrak{p} \\ &= \int_{\widehat{G_\mathfrak{p}}} \mathbf{1}_{H_\mathfrak{p}^*} \int_{H_\mathfrak{p}} d\mathfrak{a}_\mathfrak{p}\chi_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})d\chi_\mathfrak{p} \\ &= \int_{H_\mathfrak{p}} d\mathfrak{a}_\mathfrak{p} \int_{H_\mathfrak{p}^*} d\chi_\mathfrak{p} \end{aligned}$$

This is true for all $\mathfrak{a}_\mathfrak{p} \in G_\mathfrak{p}$ and hence for all $\mathfrak{a}_\mathfrak{p} \in H_\mathfrak{p}$, we have

$$\int_{H_\mathfrak{p}} d\mathfrak{a}_\mathfrak{p} \int_{H_\mathfrak{p}^*} d\chi_\mathfrak{p} = 1$$

Hence, we can conclude that $\int_{H_\mathfrak{p}^*} d\chi_\mathfrak{p} = 1$ for all but finitely many $\mathfrak{p}$. This allows us to define the measure $d\chi = \prod_\mathfrak{p} d\chi_\mathfrak{p}$.

Proof. We apply theorem 24 to $f(\mathfrak{a})\overline{\chi}(\mathfrak{a}) = \prod_\mathfrak{p} f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})\overline{\chi(\mathfrak{a}_\mathfrak{p})}$ to see that the Fourier transform of f is the product of Fourier transforms. Since $f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ is measurable function in $G_\mathfrak{p}$, therefore $\widehat{f}_\mathfrak{p}(\chi_\mathfrak{p}) \in L_1(\widehat{G_\mathfrak{p}})$ for all $\mathfrak{p}$. By the observation just made above, we see that $\widehat{f}_\mathfrak{p}(\chi_\mathfrak{p})$ is the characteristic function of $H_\mathfrak{p}^*$. Hence, $\widehat{f}(\chi)$ is also in $L_1(\widehat{G})$ and consequently f is measurable function of G. □

Corollary 26 ([Cas+76] Corollary 3.3.1)**.** *The measure $d\chi = \prod_\mathfrak{p} d\chi_\mathfrak{p}$ is dual to $d = \prod_\mathfrak{p} d\mathfrak{a}_\mathfrak{p}$*

2.2. Global Additive Theory

The analogue to $K_\mathfrak{p}^+$ in local theory will be the adeles. We have a lot of similar results as will be seen in this section.

Theorem 27 ([Cas+76] Theorem 4.1.1)**.** *$\mathbb{A}_K$ is naturally isomorphic to its character group $\widehat{\mathbb{A}_K}$ if we identify an element $\mathfrak{X} = (\cdots, \mathfrak{X}_\mathfrak{p}, \cdots)$ with the character $\mathfrak{n} \mapsto \exp(2\pi i \Lambda(\mathfrak{n}\mathfrak{X}))$ where $\Lambda(\mathfrak{X}) = \sum_\mathfrak{p} \Lambda_\mathfrak{p}(\mathfrak{X}_\mathfrak{p})$*

Proof. The character group of $\mathbb{A}_K$ is the restricted direct product of $K_\mathfrak{p}^+$ with respect to $\mathcal{O}_\mathfrak{p}^*$. And, any character of $\mathbb{A}_K$ is a tuple of local characters. The local characters that are trivial on $\mathcal{O}_\mathfrak{p}$ are precisely those for which $\Lambda_\mathfrak{p}(\xi\eta) = 0 \ \forall \xi \in \mathcal{O}_\mathfrak{p} \Leftrightarrow \lambda(\mathrm{Tr}(\xi\eta)) = 0 \ \forall \xi \in \mathcal{O}_\mathfrak{p} \Leftrightarrow \mathrm{Tr}(\xi\eta) \in \mathbb{Z}_p \ \forall \xi \in \mathcal{O}_\mathfrak{p} \Leftrightarrow \eta \in \mathfrak{d}_\mathfrak{p}^{-1}$. Therefore, the character group of $\mathbb{A}_K$ is isomorphic to the restricted direct product of $\widehat{K_\mathfrak{p}}$ with respect to $\mathfrak{d}\mathfrak{p}^{-1}$ with a typical element $\mathfrak{n} = (\cdots, \mathfrak{n}_\mathfrak{p}, \cdots)$. But for all but finitely many $\mathfrak{p}$, we have $\mathcal{O}_\mathfrak{p} = \mathfrak{d}_\mathfrak{p}$ (since only finitely many primes are ramified). This is equivalent to saying that $\mathfrak{n}_\mathfrak{p}$ is trivial for almost all $\mathfrak{p}$. This completes the proof. □

Theorem 28 ([Cas+76] Theorem 4.1.2)**.** *If $f(\mathfrak{X}) \in L_1(\mathbb{A}_K)$, we define the Fourier transform*

$$\widehat{f}(\mathfrak{y}) = \int_{\mathbb{A}_K} f(\mathfrak{X}) \exp(-2\pi i \Lambda(\mathfrak{y}\mathfrak{X}))$$

then for $f(\mathfrak{X})$ measurable, the inversion formula is

$$f(\mathfrak{X}) = \int_{\widehat{\mathbb{A}_K}} \widehat{f}(\mathfrak{y}) \exp(2\pi i \Lambda(\mathfrak{X}\mathfrak{y}))$$

Lemma 29 ([Cas+76] Lemma 4.1.1)**.** *The map $\mathfrak{X} \mapsto \mathfrak{b}\mathfrak{X}$ is an automorphism of $\mathbb{A}_K$ if and only if $\mathfrak{b}$ is an idele.*

Remark 30. *This can be seen as an analogue of 7 8*

Proof. For the map to be an automorphism, we need the existence of a $\mathfrak{a}' \in \mathbb{A}_K$ such that $\mathfrak{a}\mathfrak{a}' = (\cdots, 1, \cdots)$. This is both a necessary and sufficient condition. The condition is equivalent to saying that each $\mathfrak{a}_\mathfrak{p} \neq 0$ and $\mathfrak{a}'_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}^{-1}$. Moreover, since $\mathfrak{a}' \in \mathbb{A}_K$ we have $\mathfrak{a}'_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$. This means $|\mathfrak{a}_\mathfrak{p}|_\mathfrak{p} = 1$ for all but finitely many $\mathfrak{p}$. This completes the proof. □

Lemma 31 ([Cas+76] Lemma 4.1.2)**.** *For an idele $\mathfrak{b}$, $d(\mathfrak{b}\mathfrak{X}) = |\mathfrak{b}| d\mathfrak{X}$ with $|\mathfrak{b}| = \prod_\mathfrak{p} |\mathfrak{b}_\mathfrak{p}|_\mathfrak{p}$*

Proof. Let $\mu(\mathfrak{X}) = d(\mathfrak{b}\mathfrak{X})$. Then, multiplication by an idele being an automorphism implies that μ is a Haar measure. Hence, there is a positive constant c such that $\mu(\mathfrak{X}) = c d\mathfrak{X}$. Now, to find c we can take a convenient measurable set and compute accordingly.

Let M be a compact neighbourhood of 0 in $\mathbb{A}_K$. Then,

$$\int_M d\mathfrak{X} = \prod_{\mathfrak{p}} \int_{M_{\mathfrak{p}}} d\mathfrak{X}_{\mathfrak{p}}$$
$$\int_{\mathfrak{b}M} d\mathfrak{X} = \prod_{\mathfrak{p}} \int_{\mathfrak{b}_{\mathfrak{p}} M_{\mathfrak{p}}} d\mathfrak{X}_{\mathfrak{p}}$$
$$= \prod_{\mathfrak{p}} \int_{M_{\mathfrak{p}}} |\mathfrak{b}_{\mathfrak{p}}|_{\mathfrak{p}} d\mathfrak{X}_{\mathfrak{p}}$$
$$= |\mathfrak{b}| \int_M d\mathfrak{X}$$

□

Let $\mathbb{A}_K^\infty$ be the infinite part of $\mathbb{A}_K$, $\prod_{\mathfrak{p} \in S_\infty} K_{\mathfrak{p}}$. For any $\mathfrak{X} \in \mathbb{A}_K$, let $\mathfrak{X}^\infty = (\mathfrak{X}_{\mathfrak{p}})_{\mathfrak{p} \in S_\infty}$ denote the projection of $\mathfrak{X}$ in $\mathbb{A}_K^\infty$.

Lemma 32 ([Cas+76] Lemma 4.1.4). *If $\{\omega_1, \cdots, \omega_n\}$ be the integral basis for the ring of integers $\mathcal{O}$ over $\mathbb{Z}$. Then, $\{\omega_1^\infty, \cdots, \omega_n^\infty\}$ is a basis for the vector space $\mathbb{A}_K^\infty$ over the real numbers. The parallelotope*

$$D^\infty := \{\sum_{v=1}^{n} x_v \omega_v^\infty : 0 \leq x_v < 1\}$$

has volume $\sqrt{d}$ if measured against $d\mathfrak{X}^\infty$ where $d = \det(\omega_i^{(i)})^2$ is the absolute discriminant of the number field

Definition 33. *The additive fundamental domain $D \subseteq \mathbb{A}_K$ is the set of all $\mathfrak{X} \in \mathbb{A}_K$ such that $\mathfrak{X} \in \mathbb{A}_{K,S_\infty}$ and $\mathfrak{X}^\infty \in D^\infty$.*

Theorem 34 ([Cas+76] Theorem 4.1.3).
1. *$\mathbb{A}_K = \bigsqcup_{\xi \in K} \xi + D$*
2. *D has measure 1.*

Proof.
1. This is a consequence of approximation theorem.

2. To compute the measure of D, note that $D \subseteq \mathbb{A}_{K,S_\infty}$ and $D = D^\infty \mathbb{A}_K^{S_\infty}$. Hence,

$$\begin{aligned}\int_D d\mathfrak{X} &= \int_D d\mathfrak{X}_{S_\infty} \\ &= \int_{D^\infty \times \mathbb{A}_K^{S_\infty}} d\mathfrak{X}^\infty d\mathfrak{X}^{S_\infty} \\ &= \int_{D^\infty} d\mathfrak{X}^\infty \int_{\mathbb{A}_K^{S_\infty}} d\mathfrak{X}^{S_\infty} \\ &= \sqrt{|d|} \prod_{\mathfrak{p} \notin S_\infty} \mathbb{N}_\mathfrak{p} \mathfrak{d}_\mathfrak{p}^{-1/2}\end{aligned}$$

Since discriminant is the norm of the different ideal, and the global different is product of local differents, the value of the integral is 1. This completes the proof. □

Corollary 35 ([Cas+76] Corollary 4.1.1)**.** *K is a discrete subgroup of $\mathbb{A}_K$. The factor group $\mathbb{A}_K/K$ is compact.*

Proof. I will prove the theorem for $K = \mathbb{Q}$ first. The case of function field follows from similar arguments. To show the first claim, it suffices to show that the neighbourhood of 0 is isolated, since we can translate to any other point homeomorphically. Consider the open set

$$U = \{a \in \mathbb{A}_K : |a|_\infty < 1, |a|_v \leq 1 \text{ for } v < \infty\}$$

Now, for any $a \in K^\times \subseteq \mathbb{A}_K$ we have $||a|| = 1$ due to the product formula. Therefore, $U \cap K = \{0\}$. This completes the proof.
To show that $\mathbb{A}_K/K$ is compact, this is the strategy. Consider the set

$$U = [0,1] \times \prod_{p<\infty} \mathbb{Z}_p$$

is a compact set. If I can show that this product contains the coset representatives of $\mathbb{A}_K/K$, then $\mathbb{A}_K/K$ is the image of a compact set under a continuous map and hence compact.
Take $(a_v) \in \mathbb{A}_K$. If v is ∞ then we can find the largest integer N such that $0 \leq |a_\infty - N| < 1$ (just think of greatest integer function). And then we can choose any element c_∞ in the interval $(a_\infty - N - 1, a_\infty - N + 1)$.
Since $(a_v) \in \mathbb{A}_K$, there is a finite set S such that for $p \notin S$ we have $a_p \in \mathbb{Z}_p$. For $p \in S$ and $p \neq \infty$ we see that

$$a_p = \sum_{j=-N}^{\infty} b_j p^j$$

Then,

$$c_p = a_p - \sum_{j=-N}^{-1} b_j p^j \in \mathbb{Z}_p$$

and we can now look at the set $S \setminus \{p\}$ and continue the process to get c_p for $p \in S$. Next, consider the element (x_v) such that $x_v = a_p$ if $v \notin S$, $x_v = c_p$ if $v \in S$ and $x_\infty = c_\infty$. It is clear that $(x_v) \in U$. This completes the proof. □

Lemma 36 ([Cas+76] Lemma 4.1.5)**.** $\Lambda(\xi) = 0$ *for all* $\xi \in K$

Proof. If p is a rational prime lying below the prime $\mathfrak{p}$, then $\Lambda(\xi) = \sum_{\mathfrak{p}} \Lambda_{\mathfrak{p}}(\xi) = \sum_{\mathfrak{p}} \lambda_{\mathfrak{p}}(\mathrm{Tr}_{K_{\mathfrak{p}}/\mathbb{Q}_p}(\xi)) = \sum_p \left(\sum_{\mathfrak{p}|p} \lambda_p(\mathrm{Tr}_{K_{\mathfrak{p}}/\mathbb{Q}_p}(\xi)) \right)$. Since the global trace map $\mathrm{Tr}_{K/\mathbb{Q}}$ is the sum of local trace maps $\mathrm{Tr}_{K_{\mathfrak{p}}/\mathbb{Q}_p}$, we have $\Lambda(\xi) = \sum_p \lambda_p(\mathrm{Tr}_{K/\mathbb{Q}}(\xi))$. As $\mathrm{Tr}_{K/\mathbb{Q}}$ is a rational number, the problem amounts to showing $\sum_p \lambda_p(x) \equiv 0 \mod 1$ for rational x. Fix a rational prime $q \neq p$ and write $\sum_p \lambda_p(x)$ as

$$\begin{aligned}
\sum_p \lambda_p(x) &= \sum_{p \neq q, \infty} \lambda_p(x) + \lambda_q(x) + \lambda_\infty(x) \\
&= \sum_{p \neq q, \infty} \lambda_p(x) + (\lambda_q(x) - x)
\end{aligned}$$

Note that from the way λ_p is defined, we have $(\lambda_q(x) - x) \in \mathbb{Z}_q$ and also $\lambda_p(x)$ is a rational number with denominator a p-power.This means that the numerator has some power of q. Therefore, $\sum_p \lambda_p(x)$ is a q-adic integer. This completes the proof. □

Theorem 37 ([Cas+76] Theorem 4.1.4)**.** *If* K^* *is the set of all characters of the adele group* $\mathbb{A}_K$ *which are trivial on* K*, then* $K^* = K$

Proof. We know that $\mathbb{A}_K$ is naturally isomorphic to its character group. So, the characters that are trivial on K are precisely the ones coming from $\mathfrak{X} \in K$ such that $\Lambda(\mathfrak{X}\eta) = 0$ for all η. Since K^* is the character group of $\mathbb{A}_K/K$ and $\mathbb{A}_K/K$ is compact therefore K^* is discrete. Moreover, $K \subseteq K^*$ by the previous lemma so we can consider the quotient group K^*/K. As K^*/K is a discrete subgroup of the compact group $\mathbb{A}_K/K$, we conclude that K^*/K is a finite group. But K^* is a vector space over K and since K is an infinite field, the index $[K^* : K]$ cannot be finite unless 1. □

2.2.1. Poisson Summation formula

A function $\varphi(\mathfrak{X})$ is said to be periodic if $\varphi(\mathfrak{X} + \xi) = \varphi(\mathfrak{X})$ for all $\xi \in K$. Such periodic functions define a continuous function on the compact quotient group $\mathbb{A}_K/K$. Say

$\Phi(\mathfrak{X}+\xi) := \varphi(\mathfrak{X})$ defines φ.

Lemma 38 ([Cas+76] Lemma 4.2.1)**.** *If $\varphi(\mathfrak{X})$ is continuous and periodic, then*

$$\int_D \varphi(\mathfrak{X})d\mathfrak{X} = \int_{\mathbb{A}_K/K} \Phi(\mathfrak{X})d\mu$$

where μ is the Haar measure with respect to which the quotient group $\mathbb{A}_K/K$ has measure 1.

Lemma 39 ([Cas+76] Lemma 4.2.2)**.** *The Fourier transform $\widehat{\varphi}(\xi)$ of $\varphi(\mathfrak{X})$ continuous on $\mathbb{A}_K/K$ is defined by*

$$\widehat{\varphi}(\xi) = \int_D \varphi(\mathfrak{X})\exp(-2\pi i\Lambda(\xi\mathfrak{X}))d\mathfrak{X}$$

If $\varphi(\mathfrak{X})$ is continuous and periodic with the additional condition $\sum_{\xi\in K}|\widehat{\varphi}(\xi)| < \infty$, then

$$\varphi(\xi) = \sum_{\xi\in K} \widehat{\varphi}(\xi)\exp(2\pi i\Lambda(\xi\mathfrak{X}))$$

Proof. The boundedness of the sum $\sum_{\xi\in K}|\widehat{\varphi}(\xi)|$ implies that the Fourier transform $\sum_{\xi\in K}\widehat{\varphi}(\xi)$ is summable. This means the inversion formula holds. The expression then follows. □

Lemma 40 ([Cas+76] Lemma 4.2.3)**.** *If*

1. *$f(\mathfrak{X})$ is continuous, belongs to $L_1(\mathbb{A}_K)$*
2. *$\sum_{\eta\in K} f(\mathfrak{X}+\eta)$ is uniformly convergent for $\mathfrak{X}\in D$*

Then, for the resulting continuous, periodic function $\varphi(\mathfrak{X}) = \sum_{\eta\in K} f(\mathfrak{X}+\eta)$ satisfies

$$\widehat{\varphi}(\xi) = \widehat{f}(\xi)$$

Proof.

$$\begin{aligned}
\widehat{\varphi}(\xi) &= \int_D \varphi(\mathfrak{X}) \exp(-2\pi i \Lambda(\xi\mathfrak{X})) d\mathfrak{X} \\
&= \int \left(\sum_{\eta \in K} f(\mathfrak{X}+\eta) \exp(-2\pi i \Lambda(\xi\mathfrak{X})) \right) d\mathfrak{X} \\
&= \sum_{\eta \in K} \int_D f(\mathfrak{X}+\eta) \exp(-2\pi i \Lambda(\xi\mathfrak{X})) d\mathfrak{X} \text{ since the convergence is uniform on } D \\
&= \sum_{\eta \in K} \int_{\eta + D} f(\mathfrak{X}) \exp(-2\pi i \Lambda(\mathfrak{X}\xi - \eta\xi)) d\mathfrak{X} \\
&= \sum_{\eta \in K} \int_{\eta + D} f(\mathfrak{X}) \exp(-2\pi i \Lambda(\mathfrak{X}\xi)) d\mathfrak{X} \text{ since } \Lambda(\eta\xi) = 0 \\
&= \int_{\mathbb{A}_K} f(\mathfrak{X}) \exp(-2\pi i \Lambda(\xi\mathfrak{X})) d\mathfrak{X} \\
&= \widehat{f}(\xi)
\end{aligned}$$

□

Theorem 41 (Poisson Summation Formula)**.** *[[Cas+76] Lemma 4.2.4] If*

1. *$f(\mathfrak{X})$ is continuous, belongs to $L_1(\mathbb{A}_K)$*
2. *$\sum_{\eta \in K} f(\mathfrak{X}+\eta)$ is uniformly convergent for $\mathfrak{X} \in D$*
3. *$\sum_{\eta \in K} |\widehat{f}(\eta)|$ is convergent*

Then

$$\sum_{\eta \in K} f(\eta) = \sum_{\eta \in K} \widehat{f}(\eta)$$

Proof. The hypothesis of the last two lemmas are satisfied, hence we have $\widehat{\varphi}(\xi) = \widehat{f}(\xi)$. This in turn implies $\sum_{\eta \in K} |\widehat{\varphi}(\eta)|$ is convergent. So, we can invoke 39 to get

$$\varphi(\mathfrak{X}) = \sum_{\xi \in K} \widehat{\varphi}(\xi) \exp(2\pi i \Lambda(\mathfrak{X}\xi)) = \sum_{\xi \in K} \widehat{f}(\xi) \exp(2\pi i \Lambda(\mathfrak{X}\xi))$$

Now,

$$\varphi(0) = \sum_{\eta \in K} f(\eta) = \sum_{\eta \in K} \widehat{f}(\eta)$$

□

2.2.2. Riemann-Roch Theorem

When we replace $f(\mathfrak{X})$ with $f(\mathfrak{b}\mathfrak{X})$ where $\mathfrak{b}$ is an idele. Then, we have an analogue of the Riemann-Roch theorem

Theorem 42 (Riemann-Roch theorem). *[[Cas+76] Theorem 4.2.1] If*

1. *$f(\mathfrak{X})$ is continuous, belongs to $L_1(\mathbb{A}_K)$*
2. *$\sum_{\eta \in K} f(\mathfrak{b}(\mathfrak{X}+\eta))$ is convergent for all ideles $\mathfrak{b}$ and $\mathfrak{X} \in \mathbb{A}_K$, uniformly convergent for $\mathfrak{X} \in D$*
3. *$\sum_{\eta \in K} |\widehat{f}(\mathfrak{b}\eta)|$ is convergent for all ideles $\mathfrak{b}$*

Then

$$\sum_{\eta \in K} f(\eta \mathfrak{b}) = \frac{1}{|\mathfrak{b}|} \sum_{\eta \in K} \widehat{f}(\eta/\mathfrak{b})$$

Proof. Consider the function $g(\mathfrak{X}) = f(\mathfrak{b}\mathfrak{X})$. Then, g satisfies the hypothesis of Poisson-summation formula 41. Therefore, $\sum_{\eta \in K} g(\eta) = \sum_{\eta \in K} \widehat{g}(\eta)$. Moreover,

$$\begin{aligned} \widehat{g}(\mathfrak{X}) &= \int_{\mathbb{A}_K} f(\mathfrak{b}\mathfrak{X}) \exp(-2\pi i \Lambda(\mathfrak{y}\mathfrak{X})) d\mathfrak{y} \\ &= \frac{1}{|\mathfrak{b}|} \int_{\mathbb{A}_K} f(\mathfrak{y}) \exp(-2\pi i \Lambda(\mathfrak{X}\mathfrak{y}/\mathfrak{b})) d\mathfrak{y} \\ &= \frac{1}{|\mathfrak{b}|} \widehat{f}(\mathfrak{X}/\mathfrak{b}) \end{aligned}$$

This finishes the proof. □

Riemann-Roch Theorem (Geometric version)

This section follows [RV99] chapter 7, section §7.2.

Theorem 43 (Riemann-Roch, Geometric form). *Let K be a function field in one variable over $\mathbb{F}_q$. Then there exists an integer $g \geq 0$ (called the genus of K) and a divisor $\mathcal{D}$ of degree $2g-2$ (called the canonical divisor of K) such that*

$$l(D) - l(\mathcal{D} - D) = \deg D - g + 1$$

Some preliminaries:

A divisor of K is a formal linear combination

$$D = \sum_{\nu} n_{\nu} \nu$$

where v runs over all the places of K and $n_v \in \mathbb{Z}$ such that $n_v = 0$ for all but finitely many v. The divisors form an abelian group and is denoted by $\mathrm{Div}(K)$. The degree of a divisor $D \in \mathrm{Div}(K)$ is defined to be

$$\deg D := \sum_{v} n_v \deg v$$

where $\deg v = [\mathbb{F}_{q^v} : \mathbb{F}_q]$. Observe that $\deg(D + D') = \deg D + \deg D'$ and hence

$$\deg : \mathrm{Div}(K) \to \mathbb{Z}$$

is a group homomorphism. The kernel of this map is denoted by $\mathrm{Div}^0(K)$.

Given $f \in K^{\times}$, we can associate a divisor, called the principal divisor by setting $\mathrm{div} f = \sum_v \mathrm{ord}_v(f) v$ where $\mathrm{ord}_v(f)$ is the valuation of f at v. Since $\mathrm{ord}_v(f) \neq 0$ only for finitely many v, we note that $\mathrm{div}(f)$ is indeed a divisor, and $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$. The quotient $\mathrm{Div}(K)/\mathrm{div}(K^{\times})$ is denoted by $\mathrm{Pic}(K)$ called the Picard group of K. Elements of $\mathrm{Pic}(K)$ are called divisor classes. From the product formula we have

$$\prod_{v} |f|_v = 1$$

for $f \in K^{\times}$. But $|f|_v = q_v^{-\mathrm{ord}_v(f)} = q^{-deg v \mathrm{ord}_v(f)}$ for all v. Therefore $\deg \mathrm{div}(f) = \sum_v \mathrm{ord}_v(f) \deg v = 0 \Rightarrow \mathrm{div}(K^{\times}) \subseteq \mathrm{Div}^0(K)$. If $\mathrm{div} f = \mathrm{div} g$ for $f, g \in K^{\times}$, then $\mathrm{div}(f/g) = 0 \Rightarrow \alpha = f/g$ is an unit of $\mathcal{O}_v$ for all but finitely many v. Such an α must lie in $\mathbb{F}_q^{\times}$. We can summarise this in the following exact sequence:

$$1 \longrightarrow \mathbb{F}_q^{\times} \longrightarrow K^{\times} \xrightarrow{\mathrm{div}} \mathrm{Div}^0(K) \longrightarrow \mathrm{Pic}^0(K) \qquad 0$$

We now introduce a partial ordering on $\mathrm{Div}(K)$ as follows:

$$D = \sum_{v} n_v v \geq \sum_{v} n'_v v = D'$$

if and only if $n_v \geq n'_v$ for all v. To each divisor D, we can associate

$$L(D) = \{f \in K^\times : \mathrm{div} f \geq -D\} \cup \{0\}$$

Since $\mathrm{div} f$ has degree 0 for $f \in K^\times$, we have $L(0) = \mathbb{F}_q$. Furthermore, $L(D) = \{0\}$ if $\deg D < 0$. Notice that $L(D)$ is a vector space over $\mathbb{F}_q$ and we write $\ell(D)$ for the dimension of $L(D)$.

Lemma 44. *For any divisor D, $\ell(D) < \infty$*

Proof. First we extend our divisor map to ideles:

$$\begin{aligned} \mathbb{I}_K &\to \mathrm{Div}(K) \\ (x_v) &\mapsto \sum_v v(x_v) v \end{aligned}$$

This map is easily seen to be a surjection. Let $f = \prod_v f_v$ with $f_v = \mathbf{1}_{\mathcal{O}_v}$ for all v. Given any divisor $D = \sum_v n_v v$, we can associate an idele $x(D)$ such that $v(x(D)_v) = n_v$ for every v. Then, by construction we have for all $\gamma \in K^\times$

$$f(\gamma x(D)) = \begin{cases} 1 & \text{if } v(\gamma x(D)) \geq 0 \ \forall v \\ 0 & \text{otherwise} \end{cases}$$

Equivalently, for nonzero γ, $f(\gamma x(D)) \neq 0 \Leftrightarrow \gamma \in L(D)$. Since, f is of a "nice" form (like the ones seen in 23), we deduce that $\sum_{\gamma \in K} f(\gamma x(D))$ converges. But from our analysis, the sum is just $\# L(D) = q^{\ell(D)}$. Therefore, $\ell(D) < \infty$ □

Proof of 43. Pick a non-trivial character $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ that is trivial on K. At each v, let $\mathfrak{p}_v^{m_v}$ be the conductor of the component-character ψ_v. Since $m_v = 0$ for all but finitely many v, we get a divisor

$$\mathcal{D} = -\sum_v m_v v$$

If ψ' is another character of $\mathbb{A}_K$ such that it is trivial on K, then there exists $\alpha \in K^\times$ such that $\psi'(x) = \psi(\alpha x) \ \forall x \in \mathbb{A}_K$. If $\mathcal{D}'$ is the divisor associated to ψ', then $\mathcal{K}' = \mathcal{K} + \mathrm{div}(\alpha)$. Therefore, K is determined uniquely upto principal divisors.

$$q^{\ell(D)} = \textstyle\sum_{\gamma \in K^\times} f(\gamma x(D)) \tag{2.1}$$

$$|x(D)|^{-1} = \textstyle\prod_v q_v^{n_v} = q^{\sum_v n_v \deg v} = q^{\deg D} \tag{2.2}$$

We wish to show that

$$\sum_{\gamma \in K} \widehat{f}(\gamma x(D)^{-1}) = q^{\ell(\mathcal{D}-D)-g+1}$$

Remember that $\widehat{f_v} = (\mathbb{N}\mathfrak{p}_v^{m_v})^{1/2}\mathbf{1}_{\mathfrak{p}_v^{m_v}}$. Therefore

$$\prod_v (\mathbb{N}\mathfrak{p}_v^{m_v})^{1/2} = q^{-\deg \mathcal{D}/2} = q^{1-g}$$

For $\gamma \in K^\times$, we conclude

$$\widehat{f}(\gamma x(D)) = \begin{cases} q^{1-g} & \text{if } v(\gamma) \geq m_v + n_v \\ 0 & \text{otherwise} \end{cases}$$

This implies

$$\begin{aligned} \sum_{\gamma \in K} \widehat{f}(\gamma x(D)^{-1}) &= q^{1-g}\#\{\gamma : v(\gamma) \geq m_v + n_v\} \\ &= q^{1-g}q^{\ell(\mathcal{D}-D)} \\ &= q^{1-g+\ell(\mathcal{D}-D)} \end{aligned}$$

The result now follows from 42. □

2.3. Global Multiplicative Theory

2.3.1. Measures and multiplicative fundamental domain

Look at the continuous homomorphism of the ideles onto the multiplicative group of real numbers $\mathfrak{b} \mapsto |\mathfrak{b}| = \prod_{\mathfrak{p}} |\mathfrak{b}_{\mathfrak{p}}|_{\mathfrak{p}}$. The kernel of this map is a closed subgroup which we will denote by J and a general element of J will be denoted by $\mathfrak{h}$. We want to choose a subgroup T of $\mathbb{I}_K$ such that $\mathbb{I}_K = T \times J$ (this is an attempt to replicate $K_{\mathfrak{p}} = U_{\mathfrak{p}} \times \langle \varpi \rangle$). Let us start.

Choose an Archimedean prime $\mathfrak{p}_0$ arbitrarily and let T be the set of all elements of $\mathbb{I}_K$ such that $\mathfrak{b}_{\mathfrak{p}_0} > 0$ and $\mathfrak{b}_{\mathfrak{p}} = 1$ for $\mathfrak{p} \neq \mathfrak{p}_0$. Such an idele i s determined by its absolute value. Indeed, if t is the absolute value, then it represents either $(t, 1, \cdots)$ or $(\sqrt{t}, 1, \cdots)$ depending on whether $\mathfrak{p}_0$ is real or complex respectively. Hence, $\mathfrak{b} \mapsto |\mathfrak{b}|$ restricted to T is an isomorphism of T to $\mathbb{R}_{\geq 0}$. Moreover, $\mathfrak{b} = |\mathfrak{b}|\mathfrak{h}$ with $|\mathfrak{b}| \in T$ and $\mathfrak{h} = \mathfrak{b}|\mathfrak{b}|^{-1} \in J$.

This finishes the decomposition.

Since T is same as $\mathbb{R}_{\geq 0}$, we can give it the Lebesgue measure dt/t and choose a measure on J such that $\mathfrak{b} = d\mathfrak{h}\frac{dt}{t}$. This allows us to do the following manipulation for our computations:

$$\begin{aligned} \int_{\mathbb{I}_K} f(\mathfrak{b}) d\mathfrak{b} &= \int_0^\infty \left(\int_J f(t\mathfrak{h}) d\mathfrak{h} \right) \frac{dt}{t} \\ &= \int_J \left(\int_0^\infty f(t\mathfrak{h}) \frac{dt}{t} \right) d\mathfrak{h} \end{aligned}$$

Trying to replicate the approach in the section on Additive Global theory, we wish to define a fundamental domain for $J/K^\times$. The mapping of ideles onto ideals allows us to define the subgroup $J_{S_\infty} := J \cap \mathbb{I}_{K,S_\infty}$. Let S'_∞ be the set of Archimedean places except $\mathfrak{p}_0$. Consider the map

$$\begin{aligned} \mathbf{L} : J_{S_\infty} &\to \mathbb{R}^r \\ \mathfrak{b} &\mapsto (\log |\mathfrak{b}|_\mathfrak{p})_{\mathfrak{p} \in S'_\infty} \end{aligned}$$

where $r = r_1 + r_2 - 1$ with r_1 denoting the number of inequivalent real embeddings and r_2 the number of inequivalent complex embeddings. The map $\mathbf{L}$ is a continuous homomorphism and is surjective due to weak-approximation theorem.

$K^\times \cap J_{S_\infty}$ is the group of all elements $\epsilon \in K^\times$ which are units at all finite primes or equivalently are units in the ring of integers. The units ζ for which $\mathbf{L}(\zeta) = 0$ are the roots of unity in K and forms a finite cyclic group. Dirichlet's unit theorem says that the group of units ϵ modulo the roots of unity in K is a free abelian group of r generators. If $\{\epsilon_i\}_{i=1}^r$ is the basis for the group of units modulo roots of unity, the vectors $\mathbf{L}(\epsilon_i)$ form a basis for the r-dimensional space over the real numbers, and we can write for any $\mathfrak{h} \in J_{S_\infty}$,

$$\mathbf{L}(\mathfrak{h}) = \sum_{i=1}^r x_i \mathbf{L}(\epsilon_i)$$

with unique real numbers x_i. If

$$P := \{\sum_{i=1}^r x_i \mathbf{L}(\epsilon_i) : 0 \leq x_i < 1\}, \text{ and } Q = \{(\cdots, x_\mathfrak{p}, \cdots)_{\mathfrak{p} \in S'_\infty} : 0 \leq x_\mathfrak{p} < 1\}$$

then we have the following

Lemma 45 ([Cas+76] Lemma 4.3.1)**.**

$$\int_{\mathbf{L}^{-1}(P)} d\mathfrak{h} = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}} R$$

where $R = \pm\det(\log|\epsilon_i|_{\mathfrak{p}})_{1\le i\le r, \mathfrak{p}\in S'_\infty}$ *is called the regulator of* K*.*

Definition 46. *Let* h *be the class number of* K*, and select ideles* $\mathfrak{h}^{(1)}, \mathfrak{h}^{(2)}, \ldots, \mathfrak{h}^{(h)} \in J$ *such that the corresponding ideals* $\iota\mathfrak{h}^{(1)}, \iota\mathfrak{h}^{(2)}, \ldots, \iota\mathfrak{h}^{(h)}$ *represent the different ideal classes. Let* w *be the number of roots of unity in* K*. Let* E_0 *be the subset of all* $\mathfrak{b} \in \mathbf{L}^{-1}(P)$ *such that* $0 < \arg\mathfrak{b}_{\mathfrak{p}_0} < 2\pi/w$*. We define the multiplicative fundamental domain* E *for* $J/K^\times$ *by*

$$E = E_0\mathfrak{h}^{(1)} \cup E_0\mathfrak{h}^{(2)} \cup \cdots \cup E_0\mathfrak{h}^{(h)}$$

Theorem 47 ([Cas+76] Theorem 4.3.2)**.** 1. $J = \bigsqcup_\alpha \alpha E$ *, a disjoint union.*

2. $$\int_E d\mathfrak{h} = \frac{2^{r_1}(2\pi)^{r_2}hR}{\sqrt{|d|}w}$$

Proof. 1. Take an element $\mathfrak{h} \in J$. We will find an element β such that $\mathfrak{h} \in \alpha E$. Consider the ideal $\iota\mathfrak{h}$. It must belong to one of the equivalence classes $\iota\mathfrak{h}^{(i)}$. Then, $\iota(\mathfrak{h}/\mathfrak{h}^{(i)})$ is principal. Suppose it is $\alpha\mathcal{O}$, then $\iota(\mathfrak{h}/(\mathfrak{h}^{(i)}\alpha)) = \mathcal{O}$ and therefore $\mathfrak{h}/(\mathfrak{h}^{(i)}\alpha)$ must be in the kernel of ι that is J_{S_∞}. Using the basis, we have

$$\mathbf{L}\left(\frac{\mathfrak{h}}{\mathfrak{h}^{(i)}\alpha}\right) = \sum_{i=1}^{r} x_i \mathbf{L}(\epsilon_i)$$

where $x_i \in \mathbb{R}$. If $[\cdot]$ is the greatest integer function, then $\mathfrak{t} = \dfrac{\mathfrak{h}}{\mathfrak{h}^{(i)}\alpha}\prod_i \epsilon_i^{[x_i]}$ has image in P under the map $\mathbf{L}$. Therefore, $\mathfrak{t} \in \mathbf{L}^{-1}(P)$. We now want to take care of the argument of the $\mathfrak{p}_0$-th component. Take a root of unity with the closest argument to the argument of $\mathfrak{t}_{\mathfrak{p}_0}$ and thus $\mathfrak{t}/\zeta$ will bring us to E_0. This completes the proof.

2. The integral is $h\times$ (measure of E_0) = $\dfrac{h}{w}\times$ (measure of $\mathbf{L}^{-1}(P)$). Now, plugging in the value obtained in last lemma, we get that the value of the integral is

$$\frac{h}{w}\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}}R$$

□

Corollary 48 ([Cas+76] Corollary 4.3.1). *$K^\times$ is a discrete subgroup of J and therefore $J/K^\times$ is compact.*

Proof. Since E has non-zero measure, it has an interior in J. This proves that $K^\times$ is discrete in J. As E is also relatively compact, $J/K^\times$ is compact as well. □

Remark 49. *Since $K^\times$ is a discrete subgroup of J, therefore it is also discrete in $\mathbb{I}_K$.*

2.3.2. Characters

We will only concern ourselves with the characters of $\mathbb{I}_K$ that are trivial on $K^\times$. Such characters are called Hecke characters. So, when we say character after this point, we will have such a character in mind. Note that a quasi-character is a character on J since $|\chi(\mathfrak{b})| = 1$ for $\mathfrak{b} \in J$ since $J/K^\times$ is compact. Also, the characters of $\mathbb{I}_K$ that are trivial on J are exactly of the form $\chi(\mathfrak{a}) = |\mathfrak{a}|^s$ where $s \in \mathbb{C}$ is uniquely determined by $\chi(\mathfrak{a})$. Indeed, if $\chi(\mathfrak{a})$ is trivial on J, then $\chi(\mathfrak{a})$ depends on $|\mathfrak{a}|$ and is therefore just a character of the value group $(\mathbb{R}_{>0}, \cdot)$ given by $|\mathfrak{a}| \mapsto |\mathfrak{a}|^s$ for $s \in \mathbb{C}$.

We say two characters are equivalent if and only if they agree on the subgroup J.

For every quasi-character $\chi(\mathfrak{a})$, there exists a unique real number σ such that $|\chi(\mathfrak{a})| = |\mathfrak{a}|^\sigma$. We call such a σ the exponent of χ.

2.4. The Global ζ-function and functional equation

In the following section, $f(\mathfrak{X})$ denotes a complex-valued function on the adeles and $f(\mathfrak{b})$ its restriction to the ideles. The class $\mathfrak{Z}$ denotes functions f such that

1. $f(\mathfrak{X})$ and $\widehat{f}(\mathfrak{X})$ are both continuous, $\in L_1(\mathbb{A}_K)$
2. $\sum_{\xi \in K} f(\mathfrak{b}(\mathfrak{X} + \xi))$ and $\sum_{\xi \in K} \widehat{f}(\mathfrak{b}(\mathfrak{X} + \xi))$ are both convergent for each idele $\mathfrak{b}$ and adeles $\mathfrak{X}$. The convergence is uniform for $\mathfrak{X}$ ranging over D and $\mathfrak{b}$ ranging over any compact subset of $\mathbb{I}_K$
3. $f(\mathfrak{b})|\mathfrak{b}|^\sigma$ and $\widehat{f}(\mathfrak{b})|\mathfrak{b}|^\sigma \in L_1(\mathbb{I}_K)$ for $\sigma > 1$

Remark 50. *1. The first two conditions allows us to use Riemann-Roch theorem*

2. *The last condition allows us to define the zeta function as follows.*

Definition 51. *For each $f \in \mathfrak{Z}$, we can define the function $\zeta(f,\chi)$ on the domain of quasi-characters with exponent greater than 1 by*

$$\zeta(f,\chi) = \int f(\mathfrak{b})\chi(\mathfrak{b})d\mathfrak{b}$$

We call such a function a zeta-function of K.

Theorem 52 (Main theorem of Global theory: Analytic continuation and Functional equation)**.** *[[Cas+76] Theorem 4.4.1] By analytic continuation, we can extend the definition of any zeta function $\zeta(f,\chi)$ to the domain of all quasi-characters. The extended function is single-valued and regular, except at $\chi(\mathfrak{b}) = 1$ and $\chi(\mathfrak{b}) = |\mathfrak{b}|$ where it has simple poles with residues $-\kappa f(0)$ and $\kappa \widehat{f}(0)$ respectively. (κ is the volume of the multiplicative fundamental domain). Moreover, $\zeta(f,\chi)$ satisfies the functional equation*

$$\zeta(f,\chi) = \zeta(\widehat{f},\chi^{\vee})$$

Proof. For χ of exponent $\sigma > 1$, we have

$$\zeta(f,\chi) = \int_0^\infty \left(\int_J f(t\mathfrak{h})d\mathfrak{h}\right)\frac{dt}{t}$$

Set $\zeta_t(f,\chi) = \int_J f(t\mathfrak{h})d\mathfrak{h}$

$$= \int_0^1 \zeta_t(f,\chi)\frac{dt}{t} + \int_1^\infty \zeta_t(f,\chi)\frac{dt}{t}$$

Consider the second integral

$$\int_1^\infty \int_J f(\mathfrak{h}t)\chi(\mathfrak{h}t)d\mathfrak{h}\frac{dt}{t}$$

is simply

$$\int_{\mathfrak{b}\in\mathbb{I}_K:|\mathfrak{b}|\geq 1} f(\mathfrak{b})\chi(\mathfrak{b})d\mathfrak{b}$$

If $\sigma > 1$, then

$$\int_{\mathfrak{b}\in\mathbb{I}_K:|\mathfrak{b}|\geq 1} |f(\mathfrak{b})||\mathfrak{b}|^{\sigma} d\mathfrak{b}$$

is definitely finite due to the third condition in the hypothesis of $f \in \mathfrak{Z}$. The integral with $\sigma \leq 1$ is bounded by the integral with $\sigma > 1$ and therefore, the integral is defined for quasi-characters of all exponents. This solves half our problem. Now, we need to analyse

$$\int_0^1 \zeta_t(f,\chi)\frac{dt}{t}$$

This is done by using Riemann-Roch theorem. Let us see how to do that.

Lemma 53 ([Cas+76] Lemma *A* §4.4)**.** *For all quasi-characters χ, we have*

$$\zeta_t(f,\chi) + f(0)\int_E \chi(t\mathfrak{h})d\mathfrak{h} = \zeta_{1/t}(\hat{f},\chi^{\vee}) + \hat{f}(0)\int_E \chi(\mathfrak{h}/t)^{\vee} d\mathfrak{h}$$

Proof.

$$\begin{aligned}
\zeta_f(f,\chi) + f(0)\int_E \chi(t\mathfrak{h})d\mathfrak{h} &= \sum_{\alpha\in K^\times}\int_{\alpha E} f(t\mathfrak{h})\chi(t\mathfrak{h})d\mathfrak{h} + f(0)\int_E \chi(t\mathfrak{h})d\mathfrak{h} && (2.3)\\
&= \sum_{\alpha\in K^\times}\int_{E} f(\alpha t\mathfrak{h})\chi(t\mathfrak{h})d\mathfrak{h} + f(0)\int_E \chi(t\mathfrak{h})d\mathfrak{h} && (2.4)\\
&= \int_E\left(\sum_{\alpha\in K^\times} f(\alpha t\mathfrak{h})\right)\chi(t\mathfrak{h})d\mathfrak{h} + f(0)\int_E \chi(t\mathfrak{h})d\mathfrak{h} && (2.5)\\
&= \int_E\left(\sum_{\alpha\in K} f(\alpha t\mathfrak{h})\right)\chi(t\mathfrak{h})d\mathfrak{h} && (2.6)\\
&= \int_E\left(\sum_{\alpha\in K} \hat{f}(\alpha/t\mathfrak{h})\right)\frac{1}{|t\mathfrak{h}|}\chi(t\mathfrak{h})d\mathfrak{h} && (2.7)\\
&= \int_E\left(\sum_{\alpha\in K} \hat{f}(\alpha/t\mathfrak{h})\right)\chi\left(\frac{\mathfrak{h}}{t}\right)^{\vee} d\mathfrak{h} && (2.8)
\end{aligned}$$

(3.3) follows simply from the definition and using $J = \cup_\alpha \alpha E$. (3.3) $\to$ (3.4) follows from the substitution $\mathfrak{h} \mapsto \alpha\mathfrak{h}$ and observing that $d(\alpha\mathfrak{h}) = d\mathfrak{h}, \chi(\alpha t\mathfrak{h}) = \chi(t\mathfrak{h})$. (3.4) $\to$ (3.5) is possible due to the uniform convergence hypothesis for $f \in \mathfrak{Z}$. (3.6) $\to$ (3.7) follows from Riemann-Roch theorem. (3.7) $\to$ (3.8) follows from the substitution $\mathfrak{h} \mapsto 1/\mathfrak{h}$. The proof is hence complete. □

Lemma 54 ([Cas+76] Lemma *B* §4.4)**.**

$$\int_E \chi(t\mathfrak{h})d\mathfrak{h} = \begin{cases} \kappa t^s & \chi(\mathfrak{b}) = |\mathfrak{b}|^s \\ 0 & \chi(\mathfrak{h}) \text{ is not trivial on } J \end{cases}$$

Back to the proof of the theorem. The $\int_1^\infty$ was handled earlier. We will convert our $\int_0^1$ to $\int_1^\infty$ now.

$$\int_0^1 \zeta_t(f,\chi)\frac{dt}{t} = \int_0^1 \zeta_{1/t}(\widehat{f},\chi^\vee)\frac{dt}{t} + \left(\int_0^1 \kappa\widehat{f}(0)\left(\frac{1}{t}\right)^{1-s}\frac{dt}{t} - \int_0^1 \kappa f(0)t^s\frac{dt}{t}\right)$$

Note that due to the lemma above, the term in $(\cdots)$ is to be included only when χ is a character trivial on J. Now, make a substitution $t \to 1/t$ in the right hand side to get

$$\int_0^1 \zeta_t(f,\chi)\frac{dt}{t} = \int_1^\infty \zeta_t(\widehat{f},\chi^\vee)\frac{dt}{t} + \left(\frac{\kappa\widehat{f}(0)}{s-1} - \frac{\kappa f(0)}{s}\right)$$

This in turn implies

$$\zeta(f,\chi) = \int_1^\infty \zeta_t(f,\chi)\frac{dt}{t} + \int_1^\infty \zeta_t(\widehat{f},\chi^\vee)\frac{dt}{t} + \left(\frac{\kappa\widehat{f}(0)}{s-1} - \frac{\kappa f(0)}{s}\right)$$

Both the integrals are defined for all quasi-characters χ. Hence, we have an analytic continuation of $\zeta(f,\chi)$ to the domain of all quasi-characters. Also, if $\chi(\mathfrak{b}) = 1$ then $s = 0$ and we get the residue $-\kappa f(0)$ and if $\chi(\mathfrak{b}) = |\mathfrak{b}|$ we get $s = 1$ and hence the residue is $\kappa\widehat{f}(0)$. Moreover,

$$\zeta(\widehat{f},\chi^\vee) = \int_1^\infty \zeta_t(\widehat{f},\chi^\vee)\frac{dt}{t} + \int_1^\infty \zeta_t(\widehat{\widehat{f}},\chi^{\vee\vee})\frac{dt}{t} + \left(\frac{\kappa\widehat{f}(0)}{s-1} - \frac{\kappa f(0)}{s}\right)$$

Since,

$$\zeta(\widehat{f},\chi^\vee)\frac{dt}{t} = \int_J f(-t\mathfrak{h})\chi(t\mathfrak{h})\mathfrak{h}\frac{dt}{t} = \int_J f(t\mathfrak{h})\chi(t\mathfrak{h})\mathfrak{h}\frac{dt}{t} \tag{2.9}$$

The last equality comes from the fact that under the transformation $-t \mapsto t; dt/t \mapsto dt/t$ and $\chi(-1) = 1$ as χ is trivial on $K^\times$. All of this allows us to conclude

$$\zeta(f,\chi) = \zeta(\widehat{f},\chi^\vee)$$

□

3. The thesis as the GL_1 case of automorphic forms

This chapter will be vague and will mostly try to contrast Tate's theory to the general theory of automorphic forms. We will often seek refuge in Bump's book for the proofs of the statements. I will provide citations for the exact statements. This chapter follows [**bump1997**] §3.2, §3.3, §3.4, §3.5, and all notations here are exactly the same as Bump. I will clarify notations at the beginning of each section but in case of ambiguity, please refer to Bump.

3.1. Classical Automorphic forms and representations

Let $G = GL(2,\mathbb{R})^+$ be the group of 2×2 real matrices with positive determinant. We know that G acts on the Poincare upper half plant $\mathcal{H}$ by fractional linear transformations. Let $Z(\mathbb{R})$ be the center of group, which are just the scalar matrices and let $K = SO(2)$ be the maximal compact subgroup. Let Γ be a discrete subgroup of G, further assume that Γ actually is contained in $SL(2,\mathbb{R})$. We will assume that $-I \in \Gamma$ and $\Gamma\backslash\mathcal{H}$ has finite measure. We assume that $\Gamma\backslash\mathcal{H}$ is non-compact. Let χ be a character of Γ and ω a character of $Z(\mathbb{R})$; assume that $\chi(-1) = \omega(-1)$.

The group G acts on $C^\infty(G)$ by right translation. Let the representation be ρ defined by $(\rho(g)F)(x) = F(xg)$. There is also a derived action of the Lie algebra $\mathfrak{g}$, the action is given by

$$XF(g) = \frac{d}{dt}F(g\exp(tX))|_{t=0}, \quad F \in C^\infty(G)$$

This action of $\mathfrak{g}$ extends to an action of the universal enveloping algebra $U(\mathfrak{g})$ or its complexification $U(\mathfrak{g}_\mathbb{C})$. The center $\mathcal{Z}$ of $U(\mathfrak{g}_\mathbb{C})$ is a polynomial ring in two variables:

$\mathcal{Z} = \mathbb{C}[Z, \Delta]$ where

$$R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, H = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Delta = -\frac{1}{4}(H^2 + 2RL + 2LR)$$

The action of G on $\mathcal{H}$ by fractional linear transformations extends to an action on the boundary of $\mathcal{H}$ in the Riemann sphere, which is $\mathbb{R} \cup \{\infty\}$. A cusp of Γ is a point of $\mathbb{R} \cup \{\infty\}$ whose stabiliser in Γ contains a nontrivial unipotent matrix. The number of orbits of the cusps under the action of Γ is finite. We know that there is atleast one cusp since $\Gamma\backslash\mathcal{H}$ was assumed to be non-compact.

Definition 55. *Let $F \in C^\infty(G)$. We say that F is K-finite if the functions $\rho(k)F, k \in K$ span a finite dimensional vector space over $\mathbb{C}$, and F is contained in a finite dimensional $\mathcal{Z}$-invariant subspace.*

Definition 56. *Let $C(\Gamma\backslash G, \chi, \omega)$ be the space of continuous functions $F : G \to \mathbb{C}$ such that*

$$F(\gamma g) = \chi(\gamma)F(g); \gamma \in \Gamma, g \in G$$

and

$$F(zg) = \omega(z)F(g); z \in Z(\mathbb{R}), g \in G$$

Let $C^\infty(\Gamma\backslash G, \chi, \omega)$ be the space of smooth functions in $C(\Gamma\backslash G, \chi, \omega)$, and $C_c(\Gamma\backslash G, \chi, \omega)$ be the subspace of compactly supported functions modulo $Z(\mathbb{R})$ in $C(\Gamma\backslash G, \chi, \omega)$.

Definition 57. *An element F is called an automorphic form if $F \in C^\infty(\Gamma\backslash G, \chi, \omega)$, is K-finite, $\mathcal{Z}$-finite and there exists constants C, N such that*

$$|F(g)| < C||g||^N; g \in G$$

the inequality is called the condition of moderate growth. The space of such automorphic forms is denoted by $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$

Definition 58. *Let $F \in \mathcal{A}(\Gamma\backslash G, \chi, \omega)$. We first define cuspidality at ∞. Γ contains an element* *of the form* $\tau_r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$. *We say F is cuspidal at* ∞ *if either* $\chi(\tau_r) \neq 1$ *or*

$$\int_0^r F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0$$

Now, if a is an arbitrary cusp, we can choose a $\xi \in SL(2,\mathbb{R})$ *such that* $\xi(\infty) = a$. *Then,* $F'(g) = F(\xi g)$ *defines an element of* $L^2(\Gamma'\backslash G, \chi', \omega)$ *where* $\Gamma' = \xi^{-1}\Gamma\xi$, χ' *is a character* $\chi'(\gamma) = \chi(\xi\gamma\xi^{-1})$ *of* Γ'. *We say F is cuspidal at a if* F' *is cuspidal at* ∞ *which has been defined before.*

If F is cuspidal at all cusps a of Γ *then we say F is a cusp form. The space of cusp forms in* $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ *is denoted by* $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$.

Definition 59. *A* $(\mathfrak{g}, K)$*-module is a vector space V together with representations* π *of K and of* $\mathfrak{g}$ *subject to the conditions:*

1. *V decomposes into an algebraic direct sum of finite dimensional invariant subspaces under the action of K.*
2. *The representations of* $\mathfrak{g}$ *and K are "compatible", i.e.* $\pi(X)f = Xf = \frac{d}{dt}\pi(\exp(tX))f|_{t=0}$ *for all* $X \in \mathfrak{k}$ *and* $f \in V$.
3. $\pi(g)\pi(X)\pi(g^{-1})f = \pi(\mathrm{Ad}(g)X)f$ *is valid when* $g \in K, X \in \mathfrak{g}$

We say a $(\mathfrak{g}, K)$*-module V is admissible the isotypic component* $V(\sigma) = \{v \in V : \pi(k)v = \sigma(k)v\}$ *is finite dimensional for each character* σ *of K*

Theorem 60. *The spaces* $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ *and* $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$ *are stable under the action of* $U(\mathfrak{g}_\mathbb{C})$. *If* $f \in \mathcal{A}(\Gamma\backslash G, \chi, \omega)$, *then* $U(\mathfrak{g}_\mathbb{C})f$ *is an admissible* $(\mathfrak{g}, K)$*-module. If f satisfies the moderate growth condition, and* $D \in U(\mathfrak{g}_\mathbb{C})$, *then* Df *satisfies a similar estimate with the same constant N*

Definition 61. *Let* c, d *be positive constants. Then we define Siegel set* $\mathcal{F}_{c,d}$ *by*

$$\mathcal{F}_{c,d} := \{z = x + iy : 0 \leq x \leq c, y \geq c\}$$

Proposition 62. 1. *Let* $a_1, \ldots, a_h \in \mathbb{R} \cup \{\infty\}$ *be the representatives of the* Γ *orbits of cusps of* Γ, *and let* $\xi_i \in SL(2,\mathbb{R})$ *be chosen such that* $\xi_i(a_i) = \infty$. *If* $c, d > 0$ *are chosen* *properly, then the set*

$$\bigcup \xi_i^{-1}\mathcal{F}_{c,d}$$

contains a fundamental domain for Γ.

2. *Suppose that* ∞ *is a cusp of* Γ. *Then, if* d *is large enough,* $\mathcal{F}_{d,\infty}$ *contains a fundamental domain of* Γ.

The above is an important proposition and comes up in the proof of the following

Theorem 63 (Gelfand, Graev and Piatetski-Shapiro). *Let* $\phi \in C_c^\infty(G)$.

1. *There exists a constant* C *depending on* ϕ *such that for all* $f \in L_0^2(\Gamma\backslash G, \chi, \omega)$, *we have*

$$\sup_{g\in G} |\rho(\phi)f(g)| \leq C||f||_2$$

where

$$||f||_2 = \sqrt{\int_{G/Z(\mathbb{R})} |f(g)|^2 dg}$$

2. *The restriction of this operator to* $L_0^2(\Gamma\backslash G, \chi, \omega)$ *is a compact operator.*

This in turn lets us conclude a fundamental theorem in the theory

Theorem 64. *The space* $L_0^2(\Gamma\backslash G, \chi, \omega)$ *decomposes into a Hilbert space direct sum of subspaces that are invariant and irreducible under the right regular representation* ρ. *Let* H *be such a subspace. The* K*-finite vectors* H_f *in* H *are dense, and every* K*-finite vector is an element of* $C^\infty(\Gamma\backslash G, \chi, \omega)$. *The* K*-finite vectors form an irreducible admissible* $(\mathfrak{g}, K)$*-module contained in* $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$.

3.2. Automorphic representations of $GL(n)$

Let F be a number field, $\mathbf{A}$ its adele ring. Let $\mathbf{A}_f$ be the ring of finite adeles, which is a $(\mathfrak{g}_\infty, K_\infty)$-module where $\mathfrak{g}_\infty = \prod_{v\ inS_\infty} \mathfrak{gl}(m, F_v), K_\infty = \prod_{v\in S_\infty} K_v$. Note that $\mathbf{A} = F_\infty \mathbf{A}_f$ where F_∞ is the set of all elements (a_v) such that $a_v = 1$ for every Archimedean place v.

The group $GL(n, \mathbf{A})$ can be thought of as the restricted product of $GL(n, F_v)$ with respect to the subgroups $GL(n, \mathfrak{o}_v)$ which are the maximal compact subgroups for non-Archimedean v. The group $GL(n, \mathbf{A})$ is unimodular, i.e. the left and right invariant

Haar measures coincide. The subgroup $GL(n, F)$ is a discrete subgroup of $GL(n, \mathbf{A})$; follows from the fact that F is discrete in $\mathbf{A}$. Now, if G is an algebraic group defined over F, then we have $G(\mathbf{A}) = G(F_\infty)G(\mathbf{A}_f)$. We do not have cocompactness of $Gl(n, F)$ in $GL(n, \mathbf{A})$, so we instead work with a slightly weaker result

Proposition 65. *Let* $\mathbf{A}$ *be the adele ring of the global field* F. *Then, the quotient space* $Z(\mathbf{A})GL(n, F)\backslash GL(n, \mathbf{A}$ *has finite measure.*

This result follows from the following "strong approximation theorem":

Theorem 66 (Strong approximation theorem)**.** *Let* F *be an algebraic number field.*

1. $SL(n, F_\infty)SL(n, F)$ *is dense in* $SL(n, \mathbf{A})$.

2. *Let* K_0 *be an open compact subgroup of* $GL(n, \mathbf{A}_f)$. *If the image of* K_0 *in* $\mathbf{A}_f^\times$ *under the determinant map is* $\prod_{v \notin S_\infty \mathfrak{o}_v^\times}$, *then the cardinality of*

$$GL(n, F)GL(n, F_\infty)\backslash GL(n, A)/K_0$$

is equal to the class number of F.

Like the classical theory, we want a maximal compact subgroup of $GL(n, \mathbf{A})$. Let $K = \prod_v K_v$ where K_v is $O(n)$ if v is a real place, K_v is $U(n)$ is a complex place, and K_v is $GL(n, \mathfrak{o}_v)$ when v is non-Archimedean. It is clear that K is compact by Tychonoff theorem, in fact K is maximal among the compact subgroups of $GL(n, \mathbf{A})$ and every maximal compact subgroup is conjugate to K.

let ω be a unitary Hecke character, that is, a unitary character of $\mathbf{A}^\times/F^\times$. Let $L^2(GL(n, F)\backslash GL(n, \mathbf{A}),$ be the space of all functions ϕ on $GL(n, \mathbf{A})$ that are measurable with respect to Haar measure and satisfy $\phi(\gamma g) = \phi(g), \gamma \in GL(n, F)$, and

$$\phi\left(\begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} g\right) = \omega(z)\phi(g); z \in \mathbf{A}^\times$$

and, also satisfies the square integrability modulo center condition :

$$\int_{Z(\mathbf{A})GL(n,F)\backslash GL(n,\mathbf{A})} |\phi(g)|^2 dg < \infty$$

Definition 67. *We say $\phi \in L^2(GL(n,F)\backslash GL(n,\mathbf{A}),\omega)$ is cuspidal if the condition*

$$\int_{M_r(F)\backslash M_s(\mathbf{A})} \phi\left(\begin{pmatrix} I_r & X \\ & I_s \end{pmatrix} g\right) dX = 0$$

is true for all r, s such that $r+s=n$.
Let $L_0^2(GL(n,F)\backslash GL(n,\mathbf{A}),\omega)$ be the closed subspace of cuspidal elements of the Hilbert space $L^2(GL(n,F)\backslash GL(n,\mathbf{A}),\omega)$.

The action of $GL(n,\mathbf{A})$ on $L^2(GL(n,F)\backslash GL(n,\mathbf{A}),\omega)$ is through right translation:

$$(\rho(g)\phi)(x) = \phi(xg), \quad g, x \in GL(n,\mathbf{A})$$

The cuspidal subspace $L_0^2(GL(n,F)\backslash GL(n,\mathbf{A}),\omega)$ is invariant under the action. Our objective is to prove a theorem analogous to 64. For this, we need a theorem analogous to 63. For this, let us define the space of functions $C_c^\infty(GL(n,\mathbf{A}))$ as follows: all finite linear combinations of functions $\pi(g) = \prod_v \phi_v(g_v)$ where fore each place v, ϕ_v is an element of $C_c^\infty(GL(n,F_v))$ and for all but finitely many v, ϕ_v is the characteristic function of $GL(2,\mathfrak{o}_v)$. For $f \in L^2(GL(n,F)\backslash GL(n,\mathbf{A}),\omega)$, we define

$$(\rho(\phi)f)(g) = \int_{GL(n,\mathbf{A})} \phi(h)f(gh)dh$$

Theorem 68 (Gelfand, Graev, and Piatetski-Shapiro)**.** *Let $\phi \in C_c^\infty(GL(n,\mathbf{A}))$.*

1. *There exists a constant C depending on ϕ such that for all $f \in L_0^2(GL(n,F)\backslash GL(n,\mathbf{A}),\omega)$, we have*
$$\sup_{g\in GL(2,\mathbf{A})} |\rho(\phi)f(g)| \leq C||f||_2$$
where
$$||f||_2 = \sqrt{\int_{G/Z(\mathbb{R})} |f(g)|^2 dg}$$
2. *The restriction of this operator to $L_0^2(GL(n,F)\backslash GL(n,\mathbf{A}),\omega)$ is a compact operator.*

As a consequence,

Theorem 69. *The space $L_0^2(GL(n,F)\backslash GL(n,\mathbf{A}),\omega)$ decomposes into a Hilbert space sum of irreducible invariant subspaces.*

Definition 70. *An automorphic form with central quasi-character ω is a function on $GL(n, \mathbf{A})$ satisfying $\phi(\gamma g) = \phi(g), \gamma \in GL(n, F)$, and*

$$\phi\left(\begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} g\right) = \omega(z)\phi(g); z \in \mathbf{A}^\times$$

, and is smooth, K-finite, $\mathcal{Z}$-finite, and of moderate growth.

Now, I shall explain all these conditions:

1. For a function field F, smooth means that f is locally constant. For a number field F, smoothness means for every $g \in GL(n, \mathbf{A})$ there is a neighbourhood U of g and a smooth function f_g on $GL(n, F_\infty)$ such that for all $h \in U, f(h) = f_g(h_\infty)$ where h_∞ is the infinite part of h.

2. A function f is said to be K-finite if the right translation by elements of K gives a finite dimensional vector space.

3. If v is an Archimedean place, then the action of $\mathfrak{gl}(n, F_v)$ can be defined on the K-finite vectors by
$$(Xf)(g) = \frac{d}{dt} f(g \exp(tX))|_{t=0}$$
We can extend this action of $\mathfrak{gl}(n, F_v)$ to the universal enveloping algebra $U(\mathfrak{gl}(n, F_v))$. Let $\mathcal{Z}$ be the center of the universal enveloping algebra. Now, $\mathcal{Z}$-finiteness means that f lies in a finite dimensional vector space invariant under the action of $\mathcal{Z}$.

4. For the moderate growth condition, we have to define a height function. Note that we can embed $GL(n) \to \mathbb{A}^{n^2+1}$ via $g \mapsto (g, (\det g)^{-1})$. We define the local height on $||g_v||_v$ on $GL(n, F_v)$ by $||g||_v = \max_{1 \le i \le n^2+1} |g_i|_v$ and the global height is defined to be the product of local heights. Then, f has moderate growth if there are constants C, N such that $f(g) < C||g||^N$ for all $g \in GL(n, \mathbf{A})$.

The space of automorphic forms with central quasi-character is denoted by $\mathcal{A}(GL(n, F)\backslash GL(n, \mathbf{A}), \omega)$ and the subspace of cusp forms is denoted by $\mathcal{A}_0(GL(n, F)\backslash GL(n, \mathbf{A}), \omega)$

Remark 71. *In Tate's thesis, we note that the idele group is $\mathbf{A}^\times = GL(1, \mathbf{A})$. A central quasi-character is a character $\omega : GL(1, \mathbf{A}) \to S^1$. We have seen that such a character decomposes into $\omega\chi(\cdot)|\cdot|^s$. Since ϕ is an automorphic form such that $\phi(g) = \phi(1)\omega(g)$.*

Therefore, any automorphic form on $GL(1,\mathbf{A})$ is of the form $c\omega$ for $c \in \mathbb{C}$. Moreover, ϕ must be constant on $GL(1,\mathbf{A})$. This means it is a continuous mopping from the compact group $GL(1,F)\backslash GL(n,\mathbf{A}) \to \mathbb{C}^\times$ and hence the image must be in S^1. So, the central quasi-character ω is infact a Hecke character which is what we studied.

Definition 72. *Suppose we have an infinite family of vector spaces V_v indexed by a set Σ. For all but finitely many v, we are given nonzero $x_v^\circ \in V_v$. Let Ω be the set of all finite subsets S of Σ such that if $v \notin S$ then x_v° is defined. We can order Ω by inclusion and make it a directed set. If $S, T \in \Omega$ such that $S \subseteq T$, then we can define a homomorphism $\lambda_{S,T} : \bigotimes_{v\in S} V_v \to \bigotimes_{v\in T} V_v$ by sending x to x tensored with $\otimes_{v\in T\backslash S} x_v^\circ$. These maps form a direct family and we can define the restricted tensor product as*

$$\bigotimes_v V_v := \varinjlim \bigotimes_{v\in S} V_v$$

The above construction can be interpreted similarly as the restricted product topology that gives rise to structures like the adele rings, idele groups. Now, let us record an important theorem regarding restricted tensor product.

Theorem 73 (Tensor product theorem)**.** *let (V,π) be an irreducible admissible representation of $GL(n,\mathbf{A})$. Then there exists for each Archimedean place v of F, and irreducible admissible $(\mathfrak{g}_\infty, K_v)$-module (π_v, V_v), and for each non-Archimedean place v there exists an irreducible admissible representation (π_v, V_v) of $GL(n, F_v)$ such that for all but finitely many v, V_v contains a nonzero K_v-fixed vector ξ_v° such that π is the restricted tensor product of the representations π_v.*

I will now explain the terms "admissible , irreducible " representations of $GL(n,\mathbf{A})$.

Definition 74. *Let (π, V) be a representation of K, and let (ρ, V_ρ) be an irreducible finite dimensional representation of K. Let $V(\rho)$ be the sum of all K-submodules of V that are isomorphic to K. We call $V(\rho)$ the ρ-isotypic component of (V,π).*
Let V be a complex vector space that has an action of $(\mathfrak{g}_\infty, K_\infty)$ and $GL(n,\mathbf{A}_f)$ and represent both by the same π and the representation by (π, V). We can further assume that the two actions commute. We can also assume that every vector $v \in V$ is K-finite, and if ρ is an irreducible finite dimensional representation of K, then we say π is admissible if the ρ-isotypic part $V(\rho)$ is finite dimensional. The notion for admissible representations for $GL(n, F_v)$ is also defined similarly.
For v non-Archimedean, and (π_v, V_v) an irreducible representation, we say π_v is spherical if V_v contains a nonzero K_v-fixed vector.

We end this section by stating an important theorem which will serve as motivation in our construction of Whittaker models.

Theorem 75 (Multiplicity one). *Let (π, V) and (π', V') be two irreducible admissible subrepresentations of $\mathcal{A}_0(GL(n,F)\backslash GL(n,\mathbf{A}),\omega)$. If $\pi_v \cong \pi'_v$ for all Archimedean v and all but finitely many non-Archimedean v, then $V = V'$*

3.3. Zeta-functions attached to automorphic representations

3.3.1. Whittaker models

The last theorem's proof requires the construction of Whittaker models. The Whittaker models will be used to prove the local multiplicity one theorem which in turn will lead to the prove of the global or the normal (read as the English word normal) multiplicity one theorem for $n = 2$. Let us look at the statement of local multiplicity one theorem.

Theorem 76 (Local multiplicity one). *Let F be a non-Archimedean local field, ψ a nontrivial additive character of F, and let (π, V) be an irreducible admissible representation of $GL(2,F)$. Then, upto a constant multiple, there exists at most one linear functional Λ on V such that*

$$\Lambda\left(\pi\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\xi\right) = \psi(x)\Lambda(\xi)\ x \in F, \xi \in V$$

Definition 77. *A non-zero linear functional Λ on V satisfying the condition in the last theorem is called a Whittaker functional with respect to V.*

Notice that the local multiplicity one theorem only deals with non-Archimedean places. If one wants to prove it for a number field, one has to deal with Archimedean places as well. We have to be able to get a result that combines both Archimedean and non-Archimedean places. For this, we need to look at an equivalent form of the local multiplicity one theorem.

Theorem 78. *Let F be a non-Archimedean local field, ψ a non-trivial additive character of F, and let (π, V) be an irreducible admissible representation of $GL(2,F)$. Then there exists at most* *one space $\mathcal{W}$ of functions on $GL(2, F)$ such that if $W \in \mathcal{W}$, then*

$$W\left(\pi\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\xi\right) = \psi(x)W(\xi)\ x \in F, \xi \in V$$

and such that W is closed under right translation by elements of $GL(2, F)$, and the resulting representation is isomorphic to π.

Definition 79. *The space of functions $\mathcal{W}$ satisfying the conditions of the last theorem is called a Whittaker model for the representation (π, V) with respect to ψ.*

We want to be able to handle both Archimedean and non-Archimedean local fields. Let us see how to achieve this. Suppose F is a local field F, $G = GL(2, F)$, K the maximal compact subgroup and $\mathcal{H}_G$ the Hecke algebra which is $C_c^\infty(G)$ if F is non-Archimedean and convolution algebra of distributions on G that are K-finite and have support contained in K. There is a natural action of $\mathcal{H}_G$ on $C_c^\infty(G)$ which we denote by ϕ.
Let V be a simple admissible $\mathcal{H}_G$-module. The Whittaker model of (π, V) with respect to a fixed nontrivial additive character ψ of F is the space of all functions $\mathcal{W} : G \to \mathbb{C}$ such that

$$W\left(\pi\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\xi\right) = \psi(x)W(\xi)\ x \in F, \xi \in V$$

We assume that the functions in $\mathcal{W}$ are smooth and satisfy a growth condition: for a fixed $g \in G$, $W \in \mathcal{W}$, the function

$$W\left(\pi\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\xi\right)$$

is bounded by a polynomial in $|y|$ as $|y| \to \infty$. We also assume that there exists a vector space isomorphism $\xi \mapsto W_\xi$ of V onto $\mathcal{W}$ such that if $\xi \in V$ and $\phi \in \mathcal{H}_G$, we have

$$W_{\pi(\phi)\xi} = \rho(\phi)W_\xi$$

Now,

Proposition 80. *Let F be a local field, ψ a nontrivial additive character of F and let (π, V) be a simple admissible $\mathcal{H}_G$-module. Then, (π, V) has atmost one Whittaker model with respect to ψ.*

Proposition 81. *Let F be a local field, ψ a nontrivial additive character of F and let (π, V) be a simple admissible $\mathcal{H}_G$-module. Let $\mathcal{W}$ be a Whittaker model of (π, V) with respect to ψ, and let $\xi \mapsto W_\xi$ be the isomorphism discussed. Then, there exists a $\xi \in V$ such that $W_\xi(1) \neq 0$. If V is non-Archimedean and π is spherical, and if the conductor of ψ is the ring of integers $\mathfrak{o}$ of F, then we may take ξ to be $GL(2, \mathfrak{o})$-invariant.*

So, what we have constructed till now are Whittaker functions and Whittaker models for local fields. Now, we will define global Whittaker models.

Let F be a global field, and Σ the set of all places of F. If $v \in \Sigma$, let $\mathcal{H}_v = \mathcal{H}_{GL(2,F_v)}$ be the local Hecke algebra and let $\mathcal{H} = \mathcal{H}_{GL(2,\mathbf{A})} = \bigotimes_v \mathcal{H}_v$ be the restricted tensor product of $\mathcal{H}_v$ with respect to the spherical idempotents e_v°. Let ψ be a non-trivial additive character of $\mathbf{A}$ trivial on F. Let (π, V) be an irreducible admissible $GL(2, \mathbf{A})$-module. We write π as $\pi = \otimes_v \pi_v$ with (π_v, V_v) an irreducible admissible $\mathcal{H}_v$-module, and the tensor product is the restricted product with respect to ξ_v° where ξ_v° is spherical for all but finitely many v.
By a Whittaker model of π with respect to the non-trivial character of A/F, we mean a space of functions $\mathcal{W}$ of K-finite functions on $GL(2, \mathbf{A})$ such that

$$W\left(\pi\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\xi\right) = \psi(x)W(\xi) \; x \in F, \xi \in V$$

We assume that the functions $W \in \mathcal{W}$ are of moderate growth. We assume that the space $\mathcal{W}$ is closed under the action ρ of $\mathcal{H}$ on the K-finite functions and isomorphic as a $\mathcal{H}$-module to V. It is assumed that there exists an isomorphism $\xi \mapsto W_\xi$ of V onto $\mathcal{W}$ such that

$$W_{\pi(\phi)\xi} = \rho(\phi)W_\xi \; \phi \in \mathcal{H}, \xi \in V$$

The existence and uniqueness of such Whittaker models is given in [**bump1997**] §3.5. I will just state the existence and uniqueness theorems of global Whittaker models.

Theorem 82 (Existence of Global Whittaker models)**.** *Let F be a global field, $\mathbf{A}$ its adele ring, and let (π, V) be an automorphic cuspidal representation of $GL(2, F)$, so $V \subseteq \mathcal{A}_0(GL(2, F)\backslash GL(2, \mathbf{A}), \omega)$, where ω is a character of $\mathbf{A}^\times/F^\times$. If $\phi \in V$ and $g \in GL(2, \mathbf{A})$, let*

$$W_\phi(g) = \int_{\mathbf{A}/F} \phi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g\right)\psi(-x)dx$$

Then the space $\mathcal{W}$ of functions W_ϕ is a Whittaker model for π. We have the expansion

$$\phi(g) = \sum_{\alpha \in F^\times} W_\phi\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right)$$

Theorem 83 (Uniqueness of Global Whittaker models)**.** *Let (π, V) be an irreducible admissible representation of $GL(2, \mathbf{A})$. (π, V) has a Whittaker model $\mathcal{W}$ with respect to ψ if and only if each (π_v, V_v) has a Whittaker model $\mathcal{W}_v$ with respect to the character ψ_v of F_v. If this is the case, then $\mathcal{W}$ is unique and consists of all finite linear combinations of functions of the form $W(g) = \prod_v W_v(g_v)$ where $W_v \in \mathcal{W}_v$ and $W_v = W_v^\circ$ for almost all v where W_v° is the spherical element of $\mathcal{W}_v$, normalised so that $W_v^\circ(k_v) = 1$ for $k_v \in GL(2, \mathfrak{o}_v)$.*

3.3.2. Local and Global functional equation

Let q be the cardinality of the residue field $\mathfrak{o}/\langle\varpi\rangle$. We say

$$L(s, \pi) = (1 - \alpha_1 q^{-s})^{-1}(1 - \alpha_2 q^{-s})^{-1}$$

is the local L-function of π. Let ξ be a nonramified character of $F^\times$. We define

$$L(s, \pi, \xi) = (1 - \alpha_1\xi(\varpi)q^{-s})^{-1}(1 - \alpha_2\xi(\varpi)q^{-s})^{-1}$$

Let F be a global field, (π, V) an automorphic cuspidal representation of $GL(2, \mathbf{A})$. We assume that the central quasicharacter ω of π is unitary. Write $\pi = \otimes_v \pi_v$. Let S be a finite set of places such that if $v \notin S$, then π_v is spherical. If $v \notin S$, let $L_v(s, \pi_v)$ be the local L-function as defined in the previous paragraph. Let

$$L_S(s, \pi) = \prod_{v \notin S} L_v(s, \pi_v)$$

be the S-depleted L-function. We will use the existence and uniqueness theorems of global Whittaker models to get a functional equation for $L_S(s, \pi)$. This goes through the same process as what we went through while reading Tate's thesis. Let us see it through the following steps:

1. For $\phi \in V$, we have $\phi\begin{pmatrix} y & \\ & 1 \end{pmatrix}$ is rapidly decreasing as $|y| \to \infty$, i.e. for any $N > 0$

there is a constant C_N such that $\phi \begin{pmatrix} y & \\ & 1 \end{pmatrix} < C_N |y|^{-N}$ for sufficiently large $|y|$. Therefore, the "global zeta integral" defined as

$$Z(s, \phi) = \int_{\mathbf{A}^\times / F^\times} \phi \begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-1/2} d^\times y$$

is absolutely convergent for all values of s. We can use the existence of global Whittaker models to convert our integral to

$$Z(s, \phi) = \int_{\mathbf{A}^\times} W_\phi \begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-1/2} d^\times y$$

provided that it converges absolutely. The vector ϕ corresponds to the pure tensor $\otimes_v \phi_v$. Let $W_v \in \mathcal{W}_v$ be the element of the local Whittaker model corresponding to the vector $\phi_v \in V_v$. Then, by the Uniqueness of the Whittaker model, we have

$$W(g) = \prod_v W_v(g_v)$$

If we write the idele $y = (y_v)$ then the integrand can be written as

$$\prod_v W_v \begin{pmatrix} y_v & \\ & 1 \end{pmatrix} |y_v|^{s-1/2}$$

and hence the global zeta integral becomes the product

$$Z(s, \phi) = \prod_v Z_v(s, W_v)$$

where

$$Z_v(s, W_v) = \int_{F_v^\times} W_v \begin{pmatrix} y_v & \\ & 1 \end{pmatrix} |y_v|^{s-1/2} d^\times y_v$$

2. Now, a place v is said nonramified if v is non-Archimedean, π_v is a spherical principal series, the conductor of the additive character ψ_v is $\mathfrak{o}_v$, the vector ϕ_v is the spherical vector in the representation, and the Whittaker functional W_v is normalised so that $W_v(1) = 1$. We can show that

Proposition 84. *If v is nonramified, then for s sufficiently large, then $Z_v(s, W_v) =$*

$L_v(s, \pi_v)$

3. We can generalise these integrals by twisting with an idele class character or Hecke character ξ (or more explicitly, it is a character of $\mathbf{A}^\times/F^\times$). We can now define

$$Z(s, \phi, \xi) = \int_{\mathbf{A}^\times/F^\times} \phi \begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-1/2}\xi(y)d^\times y$$

And, by similar reasoning as before, we have

$$Z(s, \phi, \xi) = \int_{\mathbf{A}^\times} W_\phi \begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-1/2}\xi(y)d^\times y$$

and again, we can decompose this integral into a product

$$Z(s, \phi, \xi) = \prod_v Z_v(s, W_v, \xi_v)$$

where

$$Z_v(s, W_v, \xi_v) = \int_{F_v^\times} W_v \begin{pmatrix} y_v & \\ & 1 \end{pmatrix} \xi_v(y_v)|y_v|^{s-1/2}d^\times y_v$$

The thing to notice is that the global zeta integral is defined for all s, but the local zeta integrals are defined only for sufficiently large s. We wish to analytically continue this to the entire complex plane and such a continuation can be achieved as seen in the following

Theorem 85 (Local functional equation)**.** *The local zeta integral $Z_v(s, W_v, \xi_v)$ defined for* $\mathrm{Re}(s)$ *sufficiently large, has a meromorphic continuation to all of s. There exists a meromorphic function $\gamma_v(s, \pi_v, \xi_v, \psi_v)$ such that*

$$Z_v(1-s, \pi_v(w_1)W_v, \xi_v^{-1}w_v^{-1}) = \gamma_v(s, \pi_v, \xi_v, \psi_v)Z_v(s, W_v, \xi_v)$$

where $w_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

Here, $\gamma_v(s, \pi_v, \xi_v, \psi_v)$ plays a similar role to $\rho(\chi)$ in the functional equation of the local zeta function in Tate's thesis.

4. Let S be a finite set of places of the global field F. Suppose S contains all the

Archimedean places and that if $v \notin S$, then π_v is spherical. We define the S-depleted L-function

$$L_S(s, \pi) = \prod_{v \notin S} L_v(s, \pi_v)$$

If ξ is a Hecke character and ξ_v is nonramified for $v \notin S$, then we can also define

$$L_S(s, \pi, \xi) = \prod_{v \notin S} L_v(s, \pi_v, \xi_v)$$

We now state

Theorem 86 (Global functional equation)**.** *Let π be an automorphic cuspidal representation of $GL(2, \mathbf{A})$. Let ξ be a Hecke character of F, S a finite set of places of F containing the Archimedean places and if $v \notin S$, then π_v is spherical and ξ_v is nonramified and the additive character ψ_v has conductor $\mathfrak{o}_v$. Then,*

$$L_S(s, \pi, \xi) = \prod_{v \in S} \gamma_v(s, \pi_v, \xi_v, \psi_v) L_S(1 - s, \widehat{\pi}, \xi^{-1})$$

where $\widehat{\pi}$ is the contragredient representation.

We can define the local L-factors for the remaining places so that they are compatible with the ones we saw in Tate's thesis. Details can be found in [Bump]. This completes our exposition.

A. Topological Groups and Haar measure

This chapter follows [RV99] chapter 1.

Definition 87. *A topological group is a topological space G such that the two maps*

$$m : G \times G \to G$$
$$(x, y) \mapsto xy$$

$$\iota : G \to G$$
$$x \mapsto x^{-1}$$

are continuous.

Consider the map $t_a : G \to G; x \mapsto ax$. This map is clearly a homeomorphism with inverse $x \mapsto g^{-1}x$. So, we can just talk about the neighbourhood of identity, since we can talk about neighbourhood of other points by translation.

Lemma 88. *Let G be a topological group. Then,*

1. *Every neighbourhood U of identity contains a neighbourhood V of identity such that $VV \subseteq U$.*

2. *Every neighbourhood U of identity contains a symmetric neighbourhood V of identity.*

3. *If H is a subgroup of G, so is its closure.*

4. *Every open subgroup of G is also closed.*

5. *If K_1, K_2 are compact subsets of G, then so is K_1K_2.*

Theorem 89. *Let G be a topological group. Then,*

1. *G is T_1.*

2. *G is Hausdorff.*

3. *The identity e is closed in G.*

4. *Every point of G is closed.*

It is natural to ask about quotient topological groups. If H is a subgroup of G. Then the set of cosets G/H is given the quotient topology which is defined as the coarsest topology such that the projection map $\pi : G \to G/H$ is continuous. We will record important properties about quotient topology in the following

Proposition 90. *If G is a topological group and H is a subgroup. Then,*

1. *The quotient space G/H is homogeneous under G.*

2. *The canonical projection $\pi : G \to G/H$ is an open map.*

3. *The quotient space G/H is T_1 if and only if H is closed.*

4. *The quotient space G/H is open if and only if H is open. Moreover, if G is compact, then H is open if and only if G/H is finite.*

5. *If H is normal in G, then G/H is a topological group with respect to the quotient operation and quotient topology.*

6. *Let H be the closure of $\{e\}$ in G. Then H is normal in G, and the quotient group G/H is Hausdorff with respect to the quotient topology.*

Proposition 91. *Let G be a Hausdorff topological group. Then the following hold:*

1. *The product of a closed subset F and a compact subset K is closed.*

2. *If H is a compact subgroup of G, then $\pi : G \to G/H$ is a closed map.*

We will mostly concern ourselves locally compact spaces i.e. where every point has a compact neighbourhood.

Proposition 92. *Let G be a Hausdorff topological space. Then a subgroup H of G is locally compact (in the subspace topology) is moreover closed. In particular, every discrete subgroup of G is closed.*

Next, we will concern ourselves with the construction of Haar measure on locally compact groups. Let us recall some measure theory.

For a set X, a collection of subsets $\mathcal{F}$ is called σ-algebra if it has the following properties:

1. $X \in \mathcal{F}$
2. If $U \in \mathcal{F}$, then so is $U^c \in \mathcal{F}$.
3. If $U_n \in \mathcal{F}, n \geq 1$, and let $A = \bigcup_{n=1}^{\infty} A_n$. Then, $A \in \mathcal{F}$.

A set X with a σ-algebra of subsets $\mathcal{F}$ is called a measurable space. If X is a topological space, we can consider the smallest σ-algebra $\mathcal{B}$ containing all the open sets of X. The elements of $\mathcal{B}$ are called the Borel subsets of X.

A positive measure μ on an arbitrary measurable space $(X, \mathcal{F})$ is a function $\mu : \mathcal{F} \to \mathbb{R}_+ \cup \{\infty\}$ that is countably additive, that is

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

for any family $\{A_n\}$ of disjoint sets in $\mathcal{F}$. A positive measure defined on the Borel sets of a locally compact Hausdorff space X is called a Borel measure.

For a Borel measure μ, we say a Borel set M is outer regular if

$$\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open }\}$$

We say a Borel set M if

$$\mu(E) = \{\mu(K) : K \subseteq E, K \text{ compact }\}$$

Definition 93. *A Radon measure on X is a Borel measure that is finite on compact sets, outer regular on all Borel sets, and inner regular on all open sets.*

To define a Haar measure, we need to introduce another term. Let G be a group and μ a Borel measure on G. We say that μ is (left) translation invariant if for all Borel subsets

E of G we have

$$\mu(gE) = \mu(E)$$

for all $g \in G$.

Definition 94. *Let G be a locally compact topological group. Then, the left (right) Haar measure on G is a left (right)-invariant non-zero Radon measure.*

Theorem 95. *Let G be a locally compact group. Then, G admits a left Haar measure. Moreover, this measure is unique upto a positive scalar.*

We will now show that the existence of left Haar measure is equivalent to the existence of a right Haar measure.

Proposition 96. *Let G be a locally compact group with non-zero Radon measure μ. Define*

$$\mathcal{C}_c^+ = \{f \in \mathcal{C}_c : f(h) > 0 \forall h \in G, ||f||_u > 0\}$$

where $|\cdot|_u$ is the uniform or sup-norm. Then,

1. *The measure μ is a left Haar measure if and only if the measure $\mu_1(E) = \mu(E^{-1})$ is a right Haar measure on G.*
2. *The measure μ is a left Haar measure on G if and only if*

$$\int_G L_h f d\mu = \int_G f d\mu$$

for all $h \in G$ and $f \in \mathcal{C}_c(G)^+$. Here, $L_h f(g) = f(h^{-1}g)$.

3. *If μ is a left Haar measure on G, then μ is positive on all non-empty open subsets of G and*

$$\int_G f d\mu > 0$$

for all $f \in \mathcal{C}_c^+$

4. *If μ is a left Haar measure on G, then $\mu(G)$ is finite if and only if G is compact.*

B. Pontryagin Duality Theorem

This chapter follows [RV99] chapter 2 and [Fol16] chapter 4.

Suppose G is an abelian topological group written multiplicatively. Define $\widehat{G}$, the multiplicative group of continuous characters $\chi : G \to S^1$ of G. $\widehat{G}$ is called a Pontryagin dual of G. We will make this into a topological group by giving the space $\widehat{G}$ the compact-open topology as follows: suppose K is a compact subset of G, and V a neighbourhood of identity in S^1. We then define the basis for open neighbourhoods of the identity character in $\widehat{G}$ as the sets

$$W(K, V) = \{\chi \in \widehat{G} : \chi(K) \subseteq U\}$$

Clearly, this defines a topology on $\widehat{G}$ and for discrete topology, it coincides with point-wise convergence.

Let us define some important subsets that will come handy in the analysis. Consider the map

$$\varphi : \mathbb{R} \to S^1; x \mapsto e^{2\pi i x}$$

which is a continuous homomorphism with kernel $\mathbb{Z}$. If $\epsilon \in (0, 1]$, then define $N(\epsilon) \subseteq S^1$ to be $N(\epsilon) := \varphi((-\epsilon/3, \epsilon/3))$.

The main theorem in the analysis of abelian topological groups and its character group is:

Theorem 97. *Let G be an abelian topological group. Then,*

1. *A group homomorphism $\chi : G \to S^1$ is continuous, and hence a character of G, if and only if $\chi^{-1}(N(1))$ is an open neighbourhood of identity in G.*

2. *The family $\{W(K, N(1))\}_K$ is a neighbourhood base of the trivial character of $\widehat{G}$ in the compact open topology.*

3. *If G is discrete, then $\widehat{G}$ is compact.*

4. *If G is compact, then $\widehat{G}$ is discrete.*

5. *If G is locally compact, then $\widehat{G}$ is locally compact as well.*

Let us now go on to define the Fourier transforms. Let G be a locally compact abelian group. Then, G is equipped with a left and right Haar measure $d\mu$.

Definition 98. *Let $f \in L_1(G)$. Then, we can define the Fourier transform of f by*

$$\widehat{f}(\chi) :== \int_G f(y)\overline{\chi(y)}d\mu$$

for $\chi \in \widehat{G}$.

Theorem 99 (Fourier transform formula)**.** *There exists a Haar measure $d\chi$ on $\widehat{G}$ such that for all $f \in L_1^+(G)$ (continuous functions of positive type), we have*

$$f(x) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(y)d\chi$$

Remark 100. *The measure $d\chi$ is called the dual measure of $d\mu$. It is not clear why such a measure should exist. The proof of existence is given in*

Theorem 101 (Pontryagin Duality)**.** *The map $\alpha : G \mapsto \widehat{\widehat{G}}; g \mapsto (\chi \mapsto \chi(g))$ is an isomorphism of topological groups.*

C. Algebraic Number Theory

This chapter follows [Mar18] [Lan94]

C.1. Norm and Trace, Discriminant

A complex number $\alpha \in \mathbb{C}$ is said to be an algebraic integer if it is a root of a monic polynomial with coefficients in $\mathbb{Z}$. We want to say that the set of all algebraic integers in $\mathbb{C}$ form a ring. For this, we use the following

Lemma 102. *Let $\alpha \in \mathbb{C}$. The following are equivalent.*

1. *α is an algebraic integer;*
2. *The additive group of the ring $\mathbb{Z}[\alpha]$ is finitely generated;*
3. *α is a member of some subring of $\mathbb{C}$ having a finitely generated additive group;*
4. *$\alpha A \subseteq A$ for some finitely generated additive subgroup $A \subseteq \mathbb{C}$*

As a corollary of this lemma, we can conclude that if α, β are algebraic integers, then so are $\alpha + \beta$ and $\alpha\beta$. Indeed, since $\mathbb{Z}[\alpha], \mathbb{Z}[\beta]$ have finitely generated additive groups then so does $\mathbb{Z}[\alpha, \beta]$ (just multiply the generators of the two groups). But $\mathbb{Z}[\alpha, \beta]$ contain $\alpha + \beta$ and $\alpha\beta$ so by the third characterisation, they are algebraic integers. Hence, the set of algebraic integers in $\mathbb{C}$ forms a subring, denoted by say $\mathbf{A}$.

If $K/\mathbb{Q}$ is a finite extension (such fields are called number fields) of degree n. Then, there are exactly n embeddings of K into $\mathbb{C}$. Moreover, if L/K are two number fields, then we know that each embedding of K in $\mathbb{C}$ extends to exactly $[L : K]$ embeddings of L into $\mathbb{C}$. In particular, there are exactly $[L : K]$ embeddings of L into $\mathbb{C}$ that fix K pointwise.

Let K be a number field. We will define two maps on K.

Definition 103. *Let* $[K : \mathbb{Q}] = n$ *and* $\sigma_1, \sigma_2, \ldots, \sigma_n$ *be the embeddings of* K *into* $\mathbb{C}$*. Then, we have two map:*

1. *Trace map: For* $\alpha \in K$, $\mathrm{Tr}(\alpha) = \sigma_1(\alpha) + \cdots + \sigma_n(\alpha)$.
2. *Norm map: For* $\alpha \in K$, $\mathbb{N}(\alpha) = \sigma_1(\alpha) \cdots \sigma_n(\alpha)$

Clearly, Tr map is additive and $\mathbb{N}$ map is multiplicative. Moreover, we can show that $\mathrm{Tr}(\alpha), \mathbb{N}(\alpha) \in \mathbb{Q}$ for all $\alpha \in K$, and in fact, if α is an algebraic integer, then $\mathrm{Tr}(\alpha)$ and $\mathbb{N}(\alpha)$ are integers.

If L/K are number fields and $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the embeddings of L into $\mathbb{C}$ that fix K pointwise. Then, we can define the relative trace $\mathrm{Tr}_{L/K}$ and $\mathbb{N}_{L/K}$ are follows:

$$\mathrm{Tr}_{L/K}(\alpha) = \sigma_1(\alpha) + \cdots + \sigma_n(\alpha)$$

$$\mathbb{N}_{L/K}(\alpha) = \sigma_1(\alpha) \cdots \sigma_n(\alpha)$$

Like before, the relative trace is additive and relative norm map is multiplicative. Moreover, we can show that $\mathrm{Tr}_{L/K}(\alpha), \mathbb{N}_{L/K}(\alpha) \in K$ for all $\alpha \in L$, and in fact, if α is an algebraic integer in L, then $\mathrm{Tr}(\alpha)$ and $\mathbb{N}(\alpha)$ are algebraic integers in K.

We also have the important "tower law"

Theorem 104. *Let* K, L, M *be number fields such that* $K \subseteq L \subseteq M$*. Then, for all* $\alpha \in M$ *we have*

$$\mathrm{Tr}_{M/K}(\alpha) = \mathrm{Tr}_{L/K}(\mathrm{Tr}_{M/L}(\alpha)), \mathbb{N}_{M/K}(\alpha) = \mathbb{N}_{L/K}(\mathbb{N}_{M/L}(\alpha))$$

Definition 105. *Again, let* $[K : \mathbb{Q}] = n$ *and* $\sigma_1, \sigma_2, \ldots, \sigma_n$ *be the embeddings of* K *into* $\mathbb{C}$*. Then, we define the discriminant by*

$$\mathrm{disc}(\alpha_1, \ldots, \alpha_n) = \det(\sigma_i(\alpha_j))^2$$

for $\alpha_1, \ldots, \alpha_n \in K$*.*

Theorem 106.

$$\mathrm{disc}(\alpha_1, \ldots, \alpha_n) = |\mathrm{Tr}(\alpha_i \alpha_j)|$$

As a corollary, $\mathrm{disc}(\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}$ and if the α_is are algebraic integers, then the discriminant is a rational integer ($\mathbb{Z}$).

C.2. Dedekind Domains and Different ideal

Definition 107. *An integral domain R is said to be a Dedekind Domain if:*

1. *R is integrally closed in its field of fractions.*
2. *R is Noetherian.*
3. *Every non-zero prime ideal is also maximal.*

Note that "integrally closed in its field of fractions" means: if $\alpha/\beta \in \mathrm{Frac}(R)$ is a root of a monic polynomial with coefficients in R, then $\alpha/\beta \in R$. The key thing to note is that

Theorem 108. *If K is a number field, then $\mathcal{O}_K := \mathbf{A} \cap K$ is a Dedekind Domain.*

From here on, unless specified, R is a Dedekind Domain.

Theorem 109. *If $\mathfrak{a}$ is an ideal of R, then there is an ideal $\mathfrak{b}$ of R such that $\mathfrak{a}\mathfrak{b}$ is principal.*

Corollary 110.

1. *If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are ideals in R, then $\mathfrak{a}\mathfrak{c} = \mathfrak{b}\mathfrak{c}$ implies $\mathfrak{a} = \mathfrak{b}$.*
2. *In a Dedekind Domain R, $\mathfrak{a} \mid \mathfrak{b} \Leftrightarrow \mathfrak{a} \supseteq \mathfrak{b}$ for ideals $\mathfrak{a}, \mathfrak{b}$*

Using these results, we can prove that

Theorem 111. *In a Dedekind Domain R, every ideal of R can be represented as a unique product of prime ideals of R.*

Corollary 112. *If K is a number field, then the ideals of $\mathcal{O}_K$ can be factored uniquely into a product of prime ideals.*

For the discussion to follow, consider the following setup: L/K is an extension of number fields. Let $\mathcal{O}_L, \mathcal{O}_K$ be the ring of algebraic integers of L, K respectively. Capital gothic characters $\mathfrak{P}$ will denote the prime ideals in $\mathcal{O}_L$ and small gothic characters $\mathfrak{p}$ the prime ideals in $\mathcal{O}_K$.

Theorem 113. *The following are equivalent:*

1. *$\mathfrak{P} \mid \mathfrak{p}\mathcal{O}_L$;*
2. *$\mathfrak{P} \supseteq \mathfrak{p}\mathcal{O}_L$;*
3. *$\mathfrak{P} \supseteq \mathfrak{p}$;*

4. $\mathfrak{P} \cap \mathcal{O}_K = \mathfrak{p}$;

5. $\mathfrak{P} \cap K = \mathfrak{p}$

When the above equivalent conditions are satisfied, we say $\mathfrak{P}$ lies above $\mathfrak{p}$ and $\mathfrak{p}$ lies below $\mathfrak{P}$. In fact,

Theorem 114. *Every prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$ lies above a unique prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ and every prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ lies below atleast one prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$.*

Note that $\mathfrak{p}\mathcal{O}_L$ is an ideal of $\mathcal{O}_L$ and due to the theorem which says that every ideal of a Dedekind Domain can be written uniquely as a product of prime ideals, we have

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$$

The exponent e_i is called the ramification index and if $\mathfrak{P}$ is a prime ideal lying above $\mathfrak{p}$, we denote the ramification index of $\mathfrak{P}$ over $\mathfrak{p}$ by $e(\mathfrak{P}/\mathfrak{p})$.

Note that every prime ideal is also a maximal ideal in Dedekind Domain (by definition). Therefore, $\mathcal{O}_L/\mathfrak{P}$ is a field and so is $\mathcal{O}_K/\mathfrak{p}$. These are called the residue fields associated to the prime ideals $\mathfrak{P}, \mathfrak{p}$. If suppose $\mathfrak{P} \mid \mathfrak{p}$, then $\mathcal{O}_L/\mathfrak{P}$ is a field extension of $\mathcal{O}_K/\mathfrak{p}$ (we can show that the fields are finite fields). So, we can talk of degree of this extension, denoted by $f(\mathfrak{P}/\mathfrak{p})$ called the inertial degree.

Theorem 115. *Let L/K be an extension of number fields of degree n, fix a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$. We have the following formula*

$$\sum_{\mathfrak{P}|\mathfrak{p}} e(\mathfrak{P}/\mathfrak{p}) f(\mathfrak{P}/\mathfrak{p}) = n$$

A prime $\mathfrak{P}$ lying above $\mathfrak{p}$ is said to ramify if $e(\mathfrak{P}/\mathfrak{p}) > 1$. A natural question to ask is which primes are ramified and this question is answered using the different ideal which we will define now. Following that, we will see some criterions that characterise ramification and also see relation between discriminant and different ideal.

Definition 116. *Let L/K be extension of number fields, then the relative discriminant $\mathfrak{d}$ is defined as the inverse of $\mathcal{O}_L^\vee$ where*

$$\mathcal{O}_L^\vee := \{\alpha \in L : \mathrm{Tr}_{L/K}(\alpha\beta) \in \mathcal{O}_K \ \forall \ \beta \in \mathcal{O}_L\}$$

Therefore,

$$\mathfrak{d}_{L/K} := (\mathcal{O}_L^\vee)^{-1} = \{\alpha \in L : \alpha\mathcal{O}_L^\vee \subseteq \mathcal{O}_L\}$$

When $K = \mathbb{Q}$, $\mathfrak{d}_{L/\mathbb{Q}}$ is called the absolute different and this will be important. From now, if we write $\mathfrak{d}_K$, it means absolute different.

If $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$, then we define the ideal-norm of $\mathfrak{a}$ by the index $[\mathcal{O}_K : \mathfrak{a}]$, and it is denoted by $||\mathfrak{a}||$.

Theorem 117. *Let L/K be an extension of number fields of degree n.*

1. *For ideals $\mathfrak{a}, \mathfrak{b}$ in $\mathcal{O}_K$, we have*

$$||\mathfrak{a}\mathfrak{b}|| = ||\mathfrak{a}|| \cdot ||\mathfrak{b}||$$

2. *Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_K$, then*

$$||\mathfrak{a}\mathcal{O}_L|| = ||\mathfrak{a}||^n$$

3. *For $0 \neq \alpha \in \mathcal{O}_K$, we have*

$$||\langle\alpha\rangle|| = |\mathbb{N}_{K/\mathbb{Q}}(\alpha)|$$

Corollary 118. *If K is a number field, then the norm of the different ideal $\mathfrak{d}_K$ is exactly the absolute discriminant $|\mathrm{disc}(K)|$.*

Theorem 119. *The prime ideals in the different ideal $\mathfrak{d}_K$ are exactly the ones that are ramified over $\mathbb{Q}$. More explicitly, for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ lying over the prime p of $\mathbb{Q}$, with ramification index $e = e(\mathfrak{p}/p)$, the exact power of $\mathfrak{p}$ in $\mathfrak{d}_K$ is $e - 1$ if $e \not\equiv 0 \mod 4$ and $\mathfrak{p}^e \mid \mathfrak{d}_K$ otherwise.*

Corollary 120. *The prime factors of $\mathrm{disc}(K)$ are precisely the ones in $\mathbb{Q}$ that are ramified in K.*

For more details and proof of the facts about the different ideal, please refer to [Lan94]

C.3. Global and Local fields

This section follows [Cas+76][Neu99].

A global field is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$ where q is a prime power. We have already discussed global fields which are number fields. For the function field case, please refer to [Sti09]. Local fields will arise naturally as completions of global fields. We say a local field is a field K with a non-trivial metric such that the topology induced by the metric is locally compact. If the metric satisfies strong triangle inequality, then we say the local field is non-Archimedean and Archimedean otherwise. Now, let us look at a few characterisations of local fields.

Lemma 121. *Let K be a field with non-trivial absolute value. K is a local field if and only if every closed ball is compact.*

Corollary 122. *Every local field has to be complete.*

Proposition 123. *Let K be a field with a non-trivial absolute value induced by a non-trivial discrete valuation. If A is the valuation ring, and ϖ is the uniformiser, then K is a local field if and only if K is complete and the residue field $A/\varpi A$ is finite.*

Corollary 124. *If K is a global field with a non-trivial absolute value $|\cdot|_v$, then the completion K_v of K with respect to the absolute value $|\cdot|_v$ is a local field.*

Theorem 125. *Let K be a global field.*

1. *If K is Archimedean, then it is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

2. *If K is non-Archimedean, then it is isomorphic to a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q(t)$.*

Finally,

Theorem 126. *A local field K satisfies one of the following equivalent conditions:*

1. *K is $\mathbb{R}$ or $\mathbb{C}$ or the fraction field of a Discrete Valuation Ring with finite residue field.*

2. *K is isomorphic to a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q(t)$.*

3. *K is a non-discrete topological field.*

Proof. 1, 2 has been discussed. Proving 3 satisfies one of the other two can be found in [RV99] □

Two important theorems for us will be

Theorem 127 (Ostrowski's theorem)**.** *[Con] Let K be a global field. Then,*

1. *Suppose $K = \mathbb{Q}$. Then every non-trivial absolute value on K is represented by either the normal absolute value, denoted by $|\cdot|_\infty$ or a p-adic absolute value, denoted by $|\cdot|_p$.*

2. *Suppose $K = \mathbb{F}_q(t)$. Then, every non-trivial absolute value on K is represented by either the absolute value at ∞, denoted by $|\cdot|_\infty$: defined as*

$$|f/g|_\infty = q^{\deg f - \deg g}$$

or by the finite places $|\cdot|_p$ corresponding to an irreducible polynomial $p(t) \in \mathbb{F}_q[t]$.

Theorem 128 (Product formula)**.** *Let $x \in K^\times$. Then,*

$$\prod_v |x|_v = 1$$

Proof. This can be broken into two steps:

1. First suppose $K = \mathbb{Q}$ or $\mathbb{F}_q(t)$. I will prove the case of $\mathbb{Q}$, the function field case follows similarly. Say $x \in \mathbb{Q}$ is a natural number. Then, by fundamental theorem of arithmetic, $x = p_1^{e_1} \cdots p_r^{e_r}$ with $e_i > 0$. Clearly, $|x|_{p_j} = p^{-e_j}; j = 1, 2, \ldots, r$. And, $|x|_\infty = p_1^{e_1} \cdots p_r^{e_r}$. Therefore, the product

   $$\prod_{p:p \text{ prime}} |x|_p \times |x|_\infty = p_1^{-e_1} \cdots p_r^{-e_r} p_{1_1^e} \cdots p_r^{e_r} = 1$$

   Similarly, if $x \in \mathbb{Q}$, then

   $$x = \frac{p_1^{e_1} \cdots p_r^{e_r}}{q_1^{f_1} \cdots q_s^{f_s}}$$

   Clearly, $|x|_{p_i} = p_i^{-e_i}, |x|_{q_j} = q_j^{f_j}$, and $|x|_\infty = p_1^{e_1} \cdots p_r^{e_r} q_1^{-f_1} \cdots q_s^{-f_s}$ Therefore,

   $$\prod_{p:p \text{ prime}} |x|_p \times |x|_\infty = p_1^{-e_1} \cdots p_r^{-e_r} q_1^{f_1} \cdots q_s^{f_s} p_1^{e_1} \cdots p_r^{e_r} q_1^{-f_1} \cdots q_s^{-f_s} = 1$$

   This completes the proof.

2. Note that for $v|p$, we have an extension of fields $K_v/\mathbb{Q}_p$. Then, for $x \in K_v$ we have

   $$|x|_{K_v} = |\mathbb{N}_{K/\mathbb{Q}}(x)|_{\mathbb{Q}_p}$$

   Therefore, for all $x \in K^\times$ we have

   $$|x| = \prod_{p \in \{p:p \text{ prime }\} \cup \{\infty\}} \prod_{v|p} |x|_v = \prod_{p \in \{p:p \text{ prime }\} \cup \{\infty\}} \prod_{v|p} |\mathbb{N}_{K_v/\mathbb{Q}_p}(x)|_v$$

Moreover, we know that

$$K \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{v|p} K_v$$

so

$$\mathbb{N}_{L/K}(x) = \prod_{v|p} \mathbb{N}_{K_v/\mathbb{Q}_p}(x)$$

Hence, we conclude our product formula from the result for $K = \mathbb{Q}$.

□

C.4. Restricted product topology

This section follows [Sut]

We have a canonical inclusion

$$\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}} := \varinjlim \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$$

Since $\mathbb{Z}_p$ is compact, $\prod_p \mathbb{Z}_p$ is also compact by Tychonoff theorem. But, $\prod_p \mathbb{Q}_p$ is not locally compact even if $\mathbb{Q}_p$ is locally compact for all p. This fact is encapsulated in the following proposition.

Proposition 129. *Given a family $\{X_i\}_{i \in I}$ of locally compact topological spaces, the product $\prod_{i \in I} X_i$ is locally compact if and only if X_i are compact for all but finitely many i.*

To rectify this situation, we will take support of a different topology which we define next.

Definition 130. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces indexed by I and suppose $U_i \subseteq X_i$ are open sets. Then,*

$$\coprod (X_i, U_i) = \{x = (x_i) \mid \; x_i \in U_i \text{ for almost all } i \in I\}$$

is defined to be the restricted product topology. The basis for this topology is

$$\mathcal{B} = \left\{ \prod_{i \in I} V_i \text{ such that } V_i = U_i \text{ for almost all } i \in I \right\}$$

For each $j \in I$, we have a projection map

$$\pi_j : \prod (X_i, U_i) \to X_j$$

such that $(x_i) \mapsto x_j$. The map π_j is continuous since for any open set $W_j \subseteq X_j$ open, $\pi_j^{-1}(W_j)$ is the union of all basic open sets $\prod V_i \in \mathcal{B}$ with $V_j = W_j$ and thus open.

Remark 131. 1. *As sets, we always have*

$$\prod U_i \subseteq \prod (X_i, U_i) \subseteq \prod X_i$$

but in general the restricted product topology is not the same as the subspace topology as inherited from $\prod X_i$. For $\prod U_i$ is open in the restricted product topology but it is not open in the subspace topology because it does not contain the intersection of $\prod (X_i, U_i)$ with any basic open set of $\prod X_i$.

2. *The restricted product topology does not depend on the choice of U_i. Indeed,*

$$\prod (X_i, U_i) = \prod (X_i, U_i')$$

whenever $U_i = U_i'$ for almost all i. Notice that the two are identical not only as sets but also as topological spaces. Thus, it is enough to specify U_i for all but finitely many $i \in I$.

For $x \in X := \prod (X_i, U_i)$, look at the set (maybe empty)

$$S(x) = \{i \in I : x_i \notin U_i\}$$

For a finite set $S \subseteq I$, we define

$$X_S = \{x \in X : S(x) \subseteq S\} = \prod_{i \notin S} U_i \times \prod_{i \in S} X_i$$

We observe that $X_S \in \mathcal{B}$ is an open set. We can view X_S as a topological space in two ways. Both as a subspace of X and as a product of certain X_i and U_i. If we restrict $\mathcal{B}$ to a basis of the subspace X_S, we get

$$\mathcal{B}_S = \{\prod V_i : V_i \subseteq \pi_i(X_S) \text{ and } V_i = U_i = \pi_i(X_S) \text{ for almost all } i\}$$

This is the standard basis for the product topology, thus the two topologies on X_S are

identical.
Notice that if $S \subseteq T$, then $X_S \subseteq X_T$. This gives a partial order on the finite subsets $S \subseteq I$ by inclusion. The system of topological spaces $\{X_S : S \subseteq^{\text{finite}} I\}$ with the inclusion maps $\{\iota_{ST} X_S \to X_T : S \subseteq T\}$ forms a direct system. Thus, we get a direct limit

$$\varinjlim X_S := \coprod X_S / \sim$$

which is the quotient of the coproduct space modulo the equivalence relation $S \ni x \sim \iota_{ST}(x)$.

Proposition 132. *Let $\{X_i\}$ be a family of topological spaces indexed by $i \in I$, $\{U_i\}$ be a family of open sets $U_i \subseteq X_i$, and let $X := \prod(X_i, U_i)$ be the corresponding restricted product topology. Let $S \subseteq^{finite} I$ and X_S be as defined before. There is a canonical homeomorphism of topological spaces*

$$\varphi : X \to \varinjlim X_S$$

Proposition 133. *Let $\{X_i\}_{i \in I}$ be a family of locally compact topological spaces and let $U_i \subseteq X_i$ be a family of open subsets of which all but finitely many are compact. Then the restricted direct product topology $X := \prod(X_i, U_i)$ is locally compact.*

Proof. Fix a finite set $S \subseteq I$ and note that the topological space

$$X_S = \prod_{i \in S} X_i \times \prod_{i \notin S} U_i$$

This is locally compact since the product of U_i's are compact by Tychonoff's theorem, finite product of locally compact spaces is locally compact and product of locally compact and compact is locally compact. Since X_S cover X, the claim follows. For if $x \in X$ then $x \in X_S$ for some finite set $S \subseteq I$, but X_S is locally compact and therefore there is a compact neighbourhood $X_S \supseteq C \ni x$ which is also compact in X. □

C.5. Adéles

Let K be a global field (finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$). Let M_K denote the set of places of K (equivalence class of valuations). For any $v \in M_K$, by K_v we denote the completion of K at the place v, and $\mathcal{O}_v$ denotes the ring of integers of the local field K_v. It is easy to see that $\mathcal{O}_v$ is compact (closed ball in a metric space).

Definition 134. *The adele ring of K is the restricted product topology of K_v with respect to the open sets $\mathcal{O}_v$. More explicitly,*

$$\mathbb{A}_K = \{(x_v) \in \prod_{v \in M_K} K_v : x_v \in \mathcal{O}_v \textit{ for all but finitely many } v\}$$

Fix a finite set of places $S \subseteq M_K$. Then the ring of S-adeles is defined to be

$$\mathbb{A}_{K,S} := \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$$

By 132 we have $\mathbb{A}_K \simeq \varinjlim \mathbb{A}_{K,S}$ and therefore $\mathbb{A}_K$ is a topological ring. We can embed $K \hookrightarrow K_v$. Such embeddings allow us to embed $K \hookrightarrow \mathbb{A}_K$ by $x \mapsto (x, x, x, x, \dots)$. The image of K in $\mathbb{A}_K$ forms a subring called the principal adeles.

Proposition 135. *The adele ring $\mathbb{A}_K$ of a global field is locally compact and Hausdorff.*

Proof. $\mathbb{A}_K$ is locally compact due to 133. If $x \neq y \in \mathbb{A}_K$, then there exists $v \in M_K$ such that $x_v \neq y_v$. But K_v is Hausdorff, hence there are disjoint open sets $U_v \ni x_v, V_v \ni y_v$. Then $U = \prod_{v' \neq v} \mathcal{O}_v \times U_{v'}, V = \prod_{v' \neq v} \mathcal{O}_{v'} \times V_v$ are disjoint open neighbourhoods of x and y proving $\mathbb{A}_K$ is Hausdorff. □

Theorem 136 (Approximation theorem)**.** *For every global field K,*

1. $\mathbb{A}_K = K + \mathbb{A}_{K,S_\infty}$
2. $K \cap \mathbb{A}_{K,S_\infty} = \mathcal{O}_K$

Proof. We want to show for $x \in \mathbb{A}_K$ there exists $\mu \in K$ such that each component $x_v - \mu \in \mathcal{O}_v$. Our proof is in the case of number fields but essentially the same proof goes through in the case of function fields.
Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ and suppose p is the rational prime lying below it. For each $v \in M_K$, there is an integers m_v such that $\mathfrak{p}_v^{m_v} x_v \in \mathcal{O}_v$. There are only finitely many v such that $x_v \notin \mathcal{O}_v$, let that set be $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Then, we can find an integer m such that mx has all local components integral, i.e. $mx_v \in \mathcal{O}_v$ for all $v \in M_K$. Let $n_1, n_2, \dots, n_r$ be a sequence in $\mathbb{N}$ and consider the equations

$$\lambda \equiv mx_j \pmod{\mathfrak{p}_j^{n_j}}$$

where x_j is the component of x corresponding to $\mathfrak{p}_j$. Such a $\lambda \in \mathcal{O}_K$ exists due to Chinese Remainder Theorem. Let $\mu = \lambda/m$. If n_js are chosen such that it is atleast the number of times $\mathfrak{p}_j$ occurs in the factorisation of $langlem\rangle$ in $\mathcal{O}_K$, then

$$x_j - \mu = \frac{mx_j - \lambda}{m} \in \frac{\mathfrak{p}_j^{n_j}}{m}$$

By our choice of n_j we see that $x_j - \mu$ is integral for all $\mathfrak{p}_j$. At the other places, it is clear that $mx_v - \lambda \in \mathcal{O}_v$. This implies

$$x_v - \mu = \frac{mx_v - \lambda}{m} \in \frac{1}{m}\mathcal{O}_v = \mathcal{O}_v$$

And, note that $\mathcal{O}_K = \bigcap_{v \in M_K \setminus S_\infty} \mathcal{O}_v$. Hence, $K \cap \mathbb{A}_{K,S_\infty} = \mathcal{O}_K$. This completes the proof. □

C.6. Idéles

Definition 137. *Let K be a global field. Then the idele group of K is the topological group $\mathbb{I}_K$ is the restricted product topology of $K_v^\times$ with respect to $\mathcal{O}_v^\times$. Explicitly,*

$$\mathbb{I}_K = \{(x_v) \in \prod_{v \in M_K} K_v^\times : x_v \in \mathcal{O}_v^\times \textit{ for all but finitely many } v\}$$

This is a group with multiplication being componentwise. The canonical embedding $K \hookrightarrow \mathbb{A}_K$ restricts to a canonical embedding $K^\times \hookrightarrow \mathbb{I}_K$.

Consider the homomorphism

$$\begin{aligned} \iota : \mathbb{I}_K &\to \mathcal{J}_K \\ (x_v) &\mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(x_\mathfrak{p})} \end{aligned}$$

This is a surjective homomorphism and the ideal is called the ideal associated to the idele.

Proposition 138. *Let K be a global field. the idele group $\mathbb{I}_K$ is a locally compact group.*

Proof. $\mathbb{I}_K$ is Hausdorff since the topology is finer than the subspace topology of $\mathbb{A}_K^\times \subseteq \mathbb{A}_K$ which is Hausdorff as seen in previous section. The set $\mathcal{O}_v^\times$ is compact since it is a closed subspace of a compact set $\mathcal{O}$. This is true for all v finite and hence $\mathcal{O}_v$ is compact for all but finitely many v. Hence, the restricted product topology is locally compact from 132. □

Theorem 139 (Product formula)**.** *For all $x \in \mathbb{I}_K$, we have*

$$\prod_v |x|_v = 1$$

Proof. I will prove the result for number fields. We can easily adjust for function fields. $K/\mathbb{Q}$ is a finite separable extension. If p is a place of $\mathbb{Q}$, then we have the isomorphism

$$K \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{v|p} K_v$$

We therefore have

$$\mathbb{N}_{K/\mathbb{Q}}(x) = \mathbb{N}_{K\otimes\mathbb{Q}_p/\mathbb{Q}_p}(x) = \prod_{v|p} \mathbb{N}_{K_v/\mathbb{Q}_p}(x)$$

If we now consider the normalised absolute value on both sides, we have

$$|\mathbb{N}_{K/\mathbb{Q}}(x)|_p = \prod_{v|p} |\mathbb{N}_{K_v/\mathbb{Q}_p}(x)|_p = \prod_{v|p} |x|_v$$

If we now take the product over all valuations of $\mathbb{Q}$, we have

$$\prod_p |\mathbb{N}_{K/\mathbb{Q}}(x)|_p = \prod_p \prod_{v|p} |\mathbb{N}_{K_v/\mathbb{Q}_p}(x)|_p = \prod_{v \in M_K} |x|_v$$

The left most product is 1 by the product formula on $\mathbb{Q}$. This completes the proof. □

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