MASTERS THESIS

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The Gross-Stark Conjecture

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Preface

For a number field K, let $\zeta_K(s)$ denote the Dedekind zeta function, a priori defined only for $\mathrm{Re}(s) > 1$ by the Euler product

$$\prod_{\mathfrak{p}:\text{finite places}} \left(1 - \frac{1}{\mathbb{N}\mathfrak{p}^s}\right)^{-1}$$

We can analytically continue this function to the entire complex plane and obtain a functional equation as well. Dirichlet was able to show that there is a pole of $\zeta_K(s)$ at $s = 1$ and infact the residue at $s = 1$ is of utmost importance. He showed that

$$\mathrm{Res}_{s=1}\, \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}} \frac{hR}{e}$$

where r_1 is the number of real embeddings, r_2 the number of complex embeddings, d is the absolute discriminant, R is the regulator, h is the class number, e is the number of roots of unity contained in K. This is one of the many instances where the special value of a L-function is related to an arithmetic invariant of the underlying algebraic object.

Artin introduced L-functions $L(\chi, s)$ attached to any complex representation $\chi : \mathrm{Gal}(\overline{K}/K) \to \overline{\mathbb{Q}}^\times$ of the absolute Galois group of number fields. In a series of papers starting in [Sta71], Stark studied the special values of these L-functions and conjectured that

$$\mathrm{Res}_{s=0} \frac{L(\chi, s)}{s^{r_\chi}} = R(\chi)A(\chi)$$

where r_χ is given, $R(\chi)$ is the generalised regulator and $A(\chi)$ is some arithmetic constant. Stark's conjecture was refined and reformulated by Tate in [Tat84]. Soon after, Deligne-Ribet [DR80], Cassou-Nogues[Cas79], Barsky[Bar78] were able to construct p-adic L-functions which interpolate to special values of these L-functions. Gross conjectured a similar formula for the leading term of the p-dic L-functions

$$\mathrm{Res}_{s=0} \frac{L_p(\omega\chi, s)}{s^r} = R_p(\chi)A(\chi)$$

where R_p is the p-adic regulator. This conjecture is known as the Gross-Stark conjecture. Gross proved the $K = \mathbb{Q}$ case and using the methods developed in

[Wil88][Wil90], Dasgupta-Darmon-Pollack [DDP11] were able to prove the conjecture for the rank one case under the additional hypothesis that Leopoldt's conjecture holds. The assumption on Leopoldt's conjecture was removed by Ventullo in [Ven15][Ven14] and the Gross-Stark conjecture was proved in full generality by Dasgupta-Kakde-Ventullo in [DKV18]. My masters thesis is to understand the proof of the Gross-Stark conjecture in the two seminal papers.

The chapter 1 introduces us to the general Stark conjectures as contained in [Tat84]. We first translate and fill gaps in the chapters presented in Tate's book and then introduce the $\mathfrak{p}$-adic versions of Stark's conjectures as presented by Gross in [Gro81].

The cohomological interpretation of the Gross-Stark conjecture is contained in chapter 2. In the rank one case the cohomological point of view reduces the conjecture to finding a cohomology class of the appropriate type. But, the question remains very tricky in the general rank case as is explained towards the end of this chapter. We mostly follow [DDP11] for this section and also use [DKV18] for notation and few results on the orthogonality of units.

The construction of the cusp form is dealt with in chapter 3 . Firstly, we create a nice potential semi-cusp form and then act on it with some nice enough Hecke operators that transform it into a true cusp-form. Here, we also deal with the subtleties of constructing such semi-cusp form in the general rank case, where we do not have a very explicit description of the semi-cusp form as the rank one case. The construction relies crucially on some geometric inputs that is done in [Ven14], [Ven15] but is beyond the scope of this thesis. We will take Ventullo's findings at faith and proceed.

In chapter 4, the $\mathfrak{p}$-adic interpolation of the cusp forms constructed in chapter 3 is performed.

The technical heart of the paper is chapter 5 containing crucial computations that allow us to obtain the essential Hida Algebra homomorphisms which will later be used to construct the cohomology class needed to compute the Gross-Stark regulator.

The construction of the cohomology class using the methods initiated in Wiles papers [Wil86][Wil88][Wil90] (which are in turn inspired by [Rib90]) is executed in chapter 6. We will also see the interplay between the local and global bases which play an important role in determining the shape of the representation. This gives an explicit description of the cohomology class which allows us to compute the Gross-Stark regulator in the later chapter.

The final chapter 7 is where we assimilate all the information from the previous chapters and actually compute the Gross-Stark regulator using the cohomology class constructed in the previous chapter.

We try to stay as original as possible and only try to exposit the work done in [DDP11], [DKV18], [Ven15],[Ven14]. We have skipped proofs where we felt the details are sufficient (or in places where we thought it would be a tragedy to rewrite things that has already been written beautifully and probably more clearly than we could).

Notation

Let k be a global field, i.e. a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$. The places or equivalence classes of absolute values of k is denoted by $v, v', \ldots$. If $\mathbb{Q} \subseteq k$, we use $\mathfrak{p}, \mathfrak{q}, \ldots$ to denote the finite places of k to distinguish it from other ideals of the ring of integers of k (denoted by other fractal letters). Given a finite extension K/k, by $w, w', \ldots$ we denote the places of K that extend $v, v', \ldots$. We use capital gothic letters $\mathfrak{P}, \mathfrak{Q}$ to denote the places of K that divide $\mathfrak{p}, \mathfrak{q}$.

The complete local fields are denoted by $k_v, K_w, k_\mathfrak{p}, K_\mathfrak{P}$; the ring of integers by $\mathcal{O}_v, \mathcal{O}_w, \mathcal{O}_\mathfrak{p}, \mathcal{O}_\mathfrak{P}$. If w is a place of K extending v, the degree of extension $[K_w : k_v]$ is denoted by $[w : v]$.

If S is a finite set of places of k containing all the Archimedean places of k, we can define the ring of S-integers

$$\mathcal{O}_S := \{x \in k : x \in \mathcal{O}_\mathfrak{p} \ \forall\ \mathfrak{p} \notin S\} = \bigcap_{\mathfrak{p} \notin S} \mathcal{O}_\mathfrak{p}$$

to be the Dedekind domain obtained by inverting all the primes of k contained in S.

We simply write $||_v, ||_w, ||_\mathfrak{p}, ||_\mathfrak{P}, \ldots$ for the normalised absolute values attached to the places indicated in the subscript. If $x \in k^\times$, we have $\mu(xU) = |x|_v \mu(U)$ for all compact sets U in the interior of k_v and all choices of Haar measure μ on the additive group k_v. More explicitly, the absolute values are

$$|x|_v = \begin{cases} \text{usual absolute value} & \text{if } k_v \simeq \mathbb{R} \\ \text{sqaure of usual absolute value} & \text{if } k_v \simeq \mathbb{C} \\ \mathbb{N}v^{-1} & \text{if } k_v \text{ is non-Archimedean} \end{cases}$$

For $x \in \mathbb{Z}_p^\times$, we have the factorisation

$$\begin{aligned} \mathbb{Z}_p^\times &= (\mathbb{Z}/2p\mathbb{Z})^\times \times (1 + 2p\mathbb{Z}_p) \\ &= \omega(x)\langle x \rangle \end{aligned}$$

with ω and $\langle \cdot \rangle$ defined by the decomposition above.

$\langle \cdot \rangle$ will also be used to denote the ideal generated by $\cdot$. The two usage shall be clear from context.

CHAPTER 1

Introduction

This chapter (more specifically §§1.1-1.5) follows [Tat84, §1 , §§0-4] closely. I have provided proof of statements that the book chooses to leave. I claim no originality in the presentation. This chapter(§§1.1-1.5) is mostly a translation of the chapter in *loc. cit.*

1. Dirichlet's analytic class number formula

Suppose k is a number field (finite extension of $\mathbb{Q}$), and $\zeta_k(s)$ is the Dedekind zeta function of k, defined for $\mathrm{Re}(s) > 1$ by the Euler product

$$\zeta_k(s) := \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathbb{N}\mathfrak{p}^s}\right)^{-1} \tag{1}$$

where the product is over all the prime ideals of k. A famous theorem of Dedekind [Theorem 40 Mar18, p. 123], a generalisation of a theorem of Dirichlet, states that

THEOREM 1. *$\zeta_k(s)$ has a simple pole at $s = 1$, and the residue at $s = 1$ is*

$$\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}} \frac{hR}{e} \tag{2}$$

where r_1 (resp. r_2) is the number of real (resp. complex) embeddings of K, d the discriminant of k, h the class number of k, and e the number of roots of unity contained in k.

The functional equation of $\zeta_k(s)$ (appendix F) allows us to rewrite this theorem into a statement of the behaviour of $\zeta_k(s)$ around the point $s = 0$.

Proposition 2. *The Taylor expansion of $\zeta_k(s)$ around $s = 0$ is given by*

$$\zeta_k(s) = -\frac{hR}{e} s^{r_1+r_2-1} + O(s^{r_1+r_2}) \tag{3}$$

PROOF. If $\Lambda_k(s) = 2^{r_2(1-s)}|d|^{s/2}\pi^{-ns/2}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\zeta_K(s)$, then by the functional equation we have

$$\Lambda(s) = \Lambda(1-s)$$

as $W(\chi) = 1$ if χ is trivial. Thus, using Dirichlet's analytic class number formula at $s = 1$, and the fact that $\Gamma(s)$ has a pole at $s = 0$ with residue 1, we have

$$s\zeta_K(s) \sim -\frac{hR}{e} s^{r_1+r_2}$$

as s goes to 0. This completes the proof. □

The above proposition 2 gives us the first non-zero term in the Taylor series expansion of $\zeta_k(s)$ around $s = 0$. Stark's conjecture will state a similar result but for Artin L-functions. Before proceeding further, we will state Dirichlet's analytic class number formula in a slightly general setting of S-units.

Let S be a finite set of places of k containing the Archimedean places S_∞. For $\mathrm{Re}(s) > 1$, we can define the generalised zeta function

$$\zeta_{k,S}(s) = \prod_{\mathfrak{p} \notin S} \left(1 - \frac{1}{\mathbb{N}\mathfrak{p}^s}\right)^{-1} \tag{4}$$

By $Cl(\mathcal{O}_{k,S})$ we will denote the ideal class group of the S-integers, and $h_{k,S}$ will denote the size of this class group.

Definition 3. *$\mathcal{O}_S^\times$ is finitely generated abelian group and thus has a free-part and a torsion part. By the S-unit theorem (cf. the next section) the rank of the free part is $r = |S| - 1$. Let $\{u_1, \ldots, u_r\}$ be a set of fundamental units modulo the torsion $(\mathcal{O}_S^\times)_{\mathrm{tors}}$. The regulator R_S is defined to be*

$$R_S = \left| \det_{\substack{1 \le i \le r \\ v \in S \setminus \{v_0\}}} (\log |u_i|_v) \right| \tag{5}$$

where v_0 is an arbitrarily chosen Archimedean prime in S.

Remark 4. *A priori, it looks like the definition depends on the choice of the Archimedean place v_0, and the choice of basis $\{u_1, \ldots, u_r\}$. But, the dependence on v_0 can be removed by the product formula. Let $\{\epsilon_1, \ldots, \epsilon_r\}$ be another set of fundamental units. Then, one can show that*

$$R_S(\{u_i\}) = R_S(\{\epsilon_i\})[\langle u_i \rangle : \langle \epsilon_i \rangle]$$

where the index is just the determinant of the transformation matrix. For fundamental units, the determinant is 1 and thus R_S does not depend on the choice of fundamental units.

Lemma 5. *Let $\mathfrak{p}$ be a place of k not contained in S. Let $T = S \cup \{\mathfrak{p}\}$. Let m be the order of $\mathfrak{p}$ in the ideal class group of S-integers $\mathcal{O}_S$. We can conclude that*

1. $h_S = m h_T$
2. $R_T = m(\log \mathbb{N}\mathfrak{p}) R_S$
3. $\zeta_{k,T}(s) \sim (\log \mathbb{N}\mathfrak{p}) s \zeta_{k,S}(s)$ *in the neighbourhood of* $s = 0$

Proof. There is a natural map from $\mathrm{I}(\mathcal{O}_S) \to \mathrm{I}(\mathcal{O}_T)$ via $\mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_T$. This map is surjective (look at the prime factorisation). Now, combining with the standard projection map $I(\mathcal{O}_T) \to C(\mathcal{O}_T)$ we have a surjective map $C(\mathcal{O}_S) \to C(\mathcal{O}_T)$. To finish the proof of first assertion, it is enough to show the following sequence is exact:

$$0 \longrightarrow [\mathfrak{p}] \longrightarrow C(\mathcal{O}_S) \longrightarrow C(\mathcal{O}_T) \longrightarrow 0$$

where $[\mathfrak{p}]$ is the class of $\mathfrak{p}$ in the ideal class group $C(\mathcal{O}_S)$. We have already shown the surjection. Let us prove the injectivity of the map $[\mathfrak{p}] \to C(\mathcal{O}_S)$. Let $\mathfrak{a} \in I(\mathcal{O}_S)$ be in the kernel of the map $C(\mathcal{O}_S) \to C(\mathcal{O}_T)$, then there exists $\alpha \in k^\times$ such that $\mathfrak{a}\mathcal{O}_T = \alpha\mathcal{O}_T$. As $S \subseteq T$, we can conclude that $\nu_\mathfrak{q}(\mathfrak{a}) = \nu_\mathfrak{q}(\alpha\mathcal{O}_S)$ for all places $\mathfrak{q} \neq \mathfrak{p}$. Thus, $\mathfrak{a} = \mathfrak{p}^e\alpha\mathcal{O}_S$ with $e = \nu_\mathfrak{p}(\mathfrak{a}) - \nu_\mathfrak{p}(\alpha)$. As both sides are fractional ideals of $\mathcal{O}_S$ with same valuation at all places, this completes the proof of first assertion.

Let $\{u_1, \dots, u_r\}$ be a set of fundamental units of $\mathcal{O}_S^\times/(\mathcal{O}_S)_{\mathrm{tors}}$. If $\mathfrak{p}^m = \varpi\mathcal{O}_S$, then $\{u_1, \dots, u_r, \varpi\}$ is a system of units for $\mathcal{O}_T^\times/(\mathcal{O}_T)_{\mathrm{tors}}$. Indeed, if $u \in \mathcal{O}_T^\times$, then after scaling with appropriate power of ϖ we can assume that $0 \le \nu_\mathfrak{p}(u) \le m-1$. Then, $u\mathcal{O}_S = \mathfrak{p}^{\nu_\mathfrak{p}(u)}$ as the valuations of both sides is equal for all places. But the order of $[\mathfrak{p}]$ in $C(\mathcal{O}_S)$ is m and so $\nu_\mathfrak{p}(u) = 0$ or equivalently, $u \in \mathcal{O}_S^\times$. Back to the second assertion. Note that $\nu_\mathfrak{q}(\varpi) = 0$ for all $\mathfrak{q} \neq \mathfrak{p}$ and so the matrix M_T corresponding to the regulator R_T has the form

$$M_T = \left[\begin{array}{c|c} M_S & \star \\ \hline \mathbf{0} & \log|\varpi|_\mathfrak{p} \end{array}\right]$$

Hence, $R_T = R_S \log|\varpi|_\mathfrak{p} = R_S \cdot m \cdot \log \mathbb{N}\mathfrak{p}$.

The third assertion follows from the observation

$$\zeta_{k,T}(s) = (1 - \mathbb{N}\mathfrak{p}^{-s})\, \zeta_{k,S}(s)$$

and taking limit as $s \to 0$. □

The following theorem follows immediately from the above lemma.

THEOREM 6. *In the neighbourhood of $s = 0$, we have*

$$\zeta_{k,S}(s) \sim -\frac{h_S R_S}{e} s^{\#S-1}$$

2. Artin L-functions

Suppose K is now a finite Galois extension of k, with Galois group G. One has

$$\chi : G \to \mathbb{C}$$

a character of a representation $G \to GL(V)$ with V a finite dimensional vector space over $\mathbb{C}$.

Fix a finite set of places of k, S, then one can simply write

$$L_S(s,\chi) = \prod_{\mathfrak{p}\notin S} \det(1 - \sigma_{\mathfrak{P}} N\mathfrak{p}^{-s} \mid V^{I_{\mathfrak{P}}})^{-1}$$

for the Artin L-function (relative to S) attached to χ. Here $\mathfrak{P}$ denotes an arbitrary place of K lying above $\mathfrak{p}$, and $\sigma_{\mathfrak{P}} \in G_{\mathfrak{P}}/I_{\mathfrak{P}}$ is the Frobenius automorphism of the extension of the residue fields $\mathfrak{P}/\mathfrak{p}$. The function $L(s,\chi)$ does not depend on the choice of the prime $\mathfrak{P}$ as all the Frobenius elements are conjugate to each other and determinant is invariant under change of basis.

In a neighbourhood of $s = 0$ w must have

$$L_S(s,\chi) = c(\chi)s^{r(\chi)} + \mathcal{O}(s^{r(\chi)+1})$$

We are interested in finding $c(\chi)$ but first we will determine the multiplicity $r(\chi)$. Let S_K be the finite set of places of K lying above the places in S, the finite set of places of k; and Y the free abelian group with basis S_K. Let

$$X = \left\{ \sum_{w\in S_K} n_w w \in Y : \sum_{w\in S_K} n_w = 0 \right\}$$

The Galois group G acts naturally by permutation of the places w dividing v for each $v \in S$. Thus we obtain a G-module structure on Y and on X. We have an exact sequence of G-modules :

$$0 \longrightarrow X \longrightarrow Y \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$\sum n_w w \longmapsto \sum n_w$$

Definition 7 (Notation)**.** *For a $\mathbb{Z}$-module B and a subring A of $\mathbb{C}$, by AB we mean the tensor product $A \otimes_{\mathbb{Z}} B$. Let χ_X be the character of the representation $\mathbb{C}X$ of G, and similarly χ_Y of $\mathbb{C}Y$.*

Remark 8. *Note that $\chi_X = \chi_Y - 1$.*

Evidently, $\chi_Y = \bigoplus_{v\in S} \mathrm{Ind}_{G_w}^{G} \mathbf{1}_{G_w}$, where for each $v \in S$, w is a place of K dividing v chosen arbitrarily. In particular, χ_Y and χ_X take their values in $\mathbb{Z}$.

Proposition 9. *If χ is a character of a $\mathbb{C}[G]$-module V (finite dimensional $\mathbb{C}$ vector space), then*

$$r(\chi) = \left(\sum_{v\in S} \dim V^{G_w}\right) - \dim V^G = \langle \chi, \chi_X\rangle_G = \dim_{\mathbb{C}} \mathrm{Hom}_G(V^*, \mathbb{C}X) \tag{6}$$

where V^* is the dual of V.

PROOF. We have a canonical homomorphism $\mathrm{Hom}_{\mathbb{C}}(V^*, \mathbb{C}X) \simeq V^{**} \otimes_{\mathbb{C}} \mathbb{C}X \simeq V \otimes_{\mathbb{C}} \mathbb{C}X$. Thus, $\mathrm{Hom}_G(V^*, \mathbb{C}X) \simeq (V \otimes_{\mathbb{C}} \mathbb{C}X)^G$. Using the othogonality of characters one has $\dim_{\mathbb{C}} \mathrm{Hom}_G(V^*, \mathbb{C}X) = \langle \chi\chi_X, \mathbf{1}\rangle_G = \langle \chi, \overline{\chi}_X\rangle_G$. Moreover, $\chi_X = \overline{\chi}_X$ (χ_X only takes integer values) and thus we have the last equality.

The second equality follows from Frobenius reciprocity in the following way:

$$\begin{aligned}
\langle \chi, \chi_X\rangle_G &= \langle \chi, \chi_Y\rangle_G - \langle \chi, 1\rangle_G \\
&= \sum_{v \in S} \langle \chi, \mathrm{Ind}_{G_w}^G \mathbf{1}_{G_w}\rangle_G - \langle \chi, 1\rangle_G \\
&= \sum_{v \in S} \langle \chi|_{G_w}, \mathbf{1}_{G_w}\rangle_{G_w} - \dim_{\mathbb{C}} V^G \\
&= \sum_{v \in S} \dim_{\mathbb{C}} V^{G_w} - \dim_{\mathbb{C}} V^G
\end{aligned}$$

It remains to show the first equality. By Brauer-Nesbitt theorem,

$$\chi = \sum_{\psi} n_\psi \,\mathrm{Ind}_H^G \psi$$

where ψ are 1 dimensional characters of subgroups H of G. Again, by Frobenius reciprocity

$$\langle \chi, \chi_X\rangle_G = \sum n_\psi \langle \chi|_H, \psi\rangle_H$$

Next, by properties of L-functions

$$r(\chi) = \sum n_\psi r(\psi)$$

Comparing the two relations tell us that it is sufficient to study just the 1 dimensional characters ψ.

If $\chi = \mathbf{1}_G$, then $L_S(s, \chi) = \zeta_{k,S}(s)$ and so using theorem 6 gives us

$$r(\chi) = \#S - 1 = \left(\sum_{v \in S} \dim V^{G_w}\right) - \dim V^G$$

If χ is a 1-dimensional character but not the trivial character, then $V^G = \{0\}$. This handles one summand. The other summand is a bit tricky. Recall the functional equation of $L_S(s, \chi)$

$$\Lambda(1 - s, \chi) = W(\chi)\Lambda(s, \overline{\chi}) \tag{7}$$

with

$$\Lambda(s, \chi) = \Gamma_{\mathbb{R}}(s)^{a_1} \Gamma_{\mathbb{R}}\left(\frac{s+1}{2}\right)^{a_2} L(s, \chi)\Gamma_{\mathbb{C}}(s)^{r_2} \tag{8}$$

and

$$a_1 = \sum_{v \text{ real}} \dim V^{G_w}, a_2 = \sum_{v \text{ real}} \operatorname{codim} V^{G_w} \tag{9}$$

It is a well known fact that $L(s,\chi)$ does not vanish at $s = 1$ and $W(\chi)$ is a non-vanishing holomorphic function. So, if we compare order of vanishing on both sides of the functional equation, we get

$$-a_1 - r_2 + r_{S_\infty} = 0 \Leftrightarrow r_{S_\infty} = a_1 + r_2 = \sum_{v|\infty} \dim V^{G_w}$$

where the last equality comes from the fact that $\dim_{\mathbb{C}} V = 1$ and r_2 is the number of complex embeddings of k in $\overline{\mathbb{Q}}$. As

$$L_S(s,\chi) = \prod_{\substack{\mathfrak{p} \in S \setminus S_\infty \\ \chi(I_\mathfrak{p}) = 1}} (1 - \chi(\mathfrak{p}) \mathbb{N}\mathfrak{p}^{-s})\, L_{S_\infty}(s,\chi)$$

As $G_\mathfrak{p}$ is generated by $I_\mathfrak{p}$ and a Frobenius $\sigma_\mathfrak{p}$, the order of vanishing of $L_S(s,\chi)$ is exactly

$$\begin{aligned} r_S(\chi) &= \#\{\mathfrak{p} \in S \setminus S_\infty : \chi(G_\mathfrak{p}) = 1\} + r_{S_\infty} \\ &= \sum_{\mathfrak{p} \in S \setminus S_\infty} \dim V^{G_\mathfrak{p}} + r_{S_\infty} \\ &= \sum_{\mathfrak{p} \in S} \dim V^{G_\mathfrak{p}} \end{aligned}$$

This completes the proof. □

We will record the observation made in the proof as it is very crucial for our purposes.

THEOREM 10. *If χ is a 1-dimensional character of G, then*

$$r_S(\chi) = \begin{cases} \#S - 1 & \text{if } \chi = \mathbf{1}_G \\ \#\{v \in S : \chi(G_v) = 1\} & \text{otherwise} \end{cases}$$

3. Stark's regulator

We will now introduce the type of regulator attached to χ which will figure in the principal conjecture of Stark. Denote by

$$U = \{x \in K^\times : |x|_w = 1 \ \forall\, w \notin S_K\}$$

the group of S_K-units of K, and consider the logarithmic embedding

$$\lambda : U \longrightarrow \mathbb{R}X$$

$$u \mapsto \sum_{w \in S_K} \log |u|_w w$$

where X is as defined in §1.2. This is used in the proof of the theorem of S-units ([Wei95, IV–4, Theorem 9]). The kernel is the group $\mu(K)$ of roots of unity contained in K, and the image is a lattice in $\mathbb{R}X$. We shall record this as a theorem as it will be cited often.

THEOREM 11. *[Dirichlet S-unit theorem] The kernel of λ is the group of roots of unity $\mu(K)$ contained in K, and the image is a full lattice in $\mathbb{R}X$ with rank $\#S - 1$. Hence, the group $U/\mu(K)$ is a free abelian group on the $\#S - 1$ generators and $1 \otimes \lambda : \mathbb{R}U \to \mathbb{R}X$ is an isomorphism.*

On tensoring with $\mathbb{C}$, λ induces isomorphism (again called λ):

$$\mathbb{C}U \xrightarrow{\sim} \mathbb{C}X$$

compatible with the natural action of G on U and X.

This implies that the two representations of G $\mathbb{Q}U$ and $\mathbb{Q}X$ are isomorphic over $\mathbb{Q}$ (Recall that we showed the invariance of this isomorphy of finite group representations by extension of scalars (in characteristic zero) either by passing to the associated characters [Ser77, §12.1] , the note after prop. 33 or by characterising an isomorphism as a homomorphism with non-zero determinant-refer to [CF10, p. 110]).

Therefore,

$$f \colon \mathbb{Q}X \xrightarrow{\sim} \mathbb{Q}U \tag{10}$$

is an isomorphism of $\mathbb{Q}G$-module, and note again

$$f \colon \mathbb{C}X \xrightarrow{\sim} \mathbb{C}U$$

its complexification.

The automorphism $\lambda \circ f$ of $\mathbb{C}X$ induces an automorphism (functorial)

$$\mathrm{Hom}_G(V^*, \mathbb{C}X) \xrightarrow{(\lambda \circ f)_V} \mathrm{Hom}_G(V^*, \mathbb{C}X)$$

$$\varphi \longmapsto \lambda \circ f \circ \varphi$$

Recall that V^* is the dual of the vector space V and following Theorem 10, the dimension of $\mathrm{Hom}_G(V^*, \mathbb{C}X)$ is exactly $r(\chi)$.

Definition 12. *The Stark regulator attached to* f *is defined as:*

$$R(\chi, f) = \det((\lambda \circ f)_V) \tag{11}$$

It is evident that $R(\chi, f)$ *does not depend on the choice of the vector space* V *of* χ*. The choice of* f*, on the contrary, is not negligible.*

4. Stark's principal conjecture

In the notations in the previous two paragraphs, the statement of the conjecture is as follows:

Conjecture 13. *Let* $A(\chi, f) = R(\chi, f)/c(\chi) \in \mathbb{C} \in \mathbb{C}$*. Then, for all automorphisms* σ *of* $\mathbb{C}$*, one has the relation*

$$A(\chi, f)^{\alpha} = A(\chi^{\alpha}, f)$$

where $\chi^{\alpha} = \alpha \circ \chi \colon G \to \mathbb{C}$*.*

We can decompose our statement in the following manner :

(1) $A(\chi, f)$ belongs to $\mathbb{Q}(\chi)$
(2) For all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, $A(\chi, f)^{\sigma} = A(\chi^{\sigma}, f)$

Here, $\mathbb{Q}(\chi)$ is the field of values of χ. It is a cyclotomic extension, and thus Galois extension of $\mathbb{Q}$. ([Ser77, §2.1]).

It seems appropriate to reformulate the conjecture starting from the situation relative to an E (coefficient field) which allows embeddings in $\mathbb{C}$. It is, in fact sufficient to consider only number fields (finite extension of $\mathbb{Q}$).

Suppose E is a field of characteristic 0 and $\chi : G \to E$ a character of the representation $G \to \mathrm{GL}_E(V)$, where V is a vector space of finite dimension over E. (Recall that G is the Galois group of the extension K/k). Instead of assuming f is rational (as in in the previous section), let us take any G-homomorphism $f : X \to EU$.

For all $\alpha \in \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$, one can deduce from χ and V a complex character $\chi^{\alpha} = \alpha \circ \chi$ of G and its complexification $V^{\alpha} = V \otimes_{E,\alpha} \mathbb{C}$, to which 2.3 applies. In particular for each α, we can associate a L-function $L(s, \chi^{\alpha})$. Moreover, $f^{\alpha} \colon \mathbb{C} \to \mathbb{C}U$ is defined by $\mathbb{C}$-linearity from $(\alpha \circ 1) \circ f : X \to \mathbb{C}U$, and induces the endomorphism $(\lambda \circ f^{\alpha})_{V^{\alpha}}$ of $\mathrm{Hom}_G(V^{\alpha *}, \mathbb{C}X)$. Denote by $R(\chi^{\alpha}, f^{\alpha})$ its determinant (it is independent of the vector space V over E associated to χ).

In this context, we are then led to the

Conjecture 14. *There exists an element* $A(\chi, f)$ *of* E *such that, for all* $\alpha : E \to \mathbb{C}$, *we have*

$$R(\chi^\alpha, f^\alpha) = A(\chi, f)^\alpha \cdot c(\chi^\alpha) \tag{12}$$

Remark: The complex conjugation being continuous, it is easy to see that $\overline{A(\chi, f)} = A(\overline{\chi}, f)$.

4.1. Changing the isomorphism f.

Proposition 15. *The conjecture 13 implies conjecture 14.*

It is clear that one can always, in conjecture 14, one can reduce to the case $E = \mathbb{C}$ and fix an arbitrary embedding $\alpha : E \to \mathbb{C}$. It is sufficient to show the independence of choice of f in this case to show that conjecture 13 implies conjecture 14 :

Lemma 16. *If the statement in conjecture 14, with* $E = \mathbb{C}$, *is true for a particular choice of isomorphism* $f_0 : \mathbb{C}X \xrightarrow{\sim} \mathbb{C}U$, *it is also true for all* $f : X \to \mathbb{C}U$.

Proof. For each $\mathbb{C}[G]$-endomorphism θ of $\mathbb{C}X$, write $\delta(\chi, \theta)$ for the determinant of the endomorphism θ_V of $\mathrm{Hom}_G(V^*, \mathbb{C})$ induced by θ. In fact, δ is clearly independent of the choice of V associated to χ. One has:

$$R(\chi, f) = \delta(\chi, \lambda \circ f)$$

The determinants δ obeys the following results:

1. $\delta(\chi + \chi', \theta) = \delta(\chi, \theta) + \delta(\chi', \theta)$
2. $\delta(\mathrm{Ind}\, \chi, \theta) = \delta(\chi, \theta)$
3. $\delta(\mathrm{Infl} \chi, \theta) = \delta(\chi, \theta|_{\mathbb{C}X} H)$
4. $\delta(\chi, \theta\theta') = \delta(\chi, \theta)\delta(\chi, \theta')$
5. $\delta(\chi, f)^\alpha = \delta(\chi^\alpha, \theta^\alpha)$ for all $\alpha \in \mathrm{Aut}(\mathbb{C})$

Here, item 1 is trivial, item 2 follows from the fact that for all representation W of the subgroup H of G and for all $\mathbb{C}[G]$-module Z, there is a natural isomorphism $\mathrm{Hom}_G(\mathrm{Ind}_H^G W) \simeq \mathrm{Hom}_H(W, Z)$, where, in the term on the right, Z is considered a H-module. item 3 refers to the following situation:

Suppose $k \subseteq K' \subseteq K$ with K'/k Galois. Denote by H, the group $\mathrm{Gal}(K/K')$ and X' the abelian group relative to K'. We then embed X' in X by $w' = \sum_{w'|w}[w : w']w = \sum_{h \in H} w_0^h$ where $[w : w']$ is the degree of the local extension $K_w/K'_{w'}$, and w_0 is an arbitrary place of K lying above w. It is this normalisation that makes the following diagram commutative :

$$\begin{array}{ccc} U & \xrightarrow{\lambda} & \mathbb{R}X \\ \uparrow & & \uparrow \\ U' & \xrightarrow{\lambda'} & \mathbb{R}X' \end{array}$$

where the maps λ, λ' is as defined in §1.3. We then find that $X' = N_H X$ where $N_H = \sum_{h \in H} h \in \mathbb{Z}[G]$, but not, in general $X' = X^H$. Nevertheless, $N_H X$ has finite index in X^H, and thus we have $EX' = EX^H$ for a field E of characteristic 0.

That being said, item 3 is evident, the formula item 4 is trivial, as for item 5, let $\alpha : \mathbb{C} \to \mathbb{C}$ be be an embedding and write $\theta^\alpha = 1 \otimes_\alpha \theta \colon \mathbb{C} \otimes_\alpha \mathbb{C}X \to \mathbb{C} \otimes_\alpha \mathbb{C}X$. X^α is viewed as $\mathbb{C} \otimes_\alpha V$ by the usual identification

$$\mathrm{Hom}_{\mathbb{C}\otimes_\alpha \mathbb{C}[G]}(\mathbb{C} \otimes_\alpha V^*, \mathbb{C} \otimes_\alpha \mathbb{C}X) = \mathbb{C} \otimes_\alpha \mathrm{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X) \tag{13}$$

The endomorphism $(\theta^\alpha)_V$ becomes $1 \otimes_\alpha \theta_V$, and the determinant is $(\det \theta_V)^\alpha$.

The statement of the lemma now follows from item 5 and the obvious relation:

$$A(\chi, f) = A(\chi, f_0)\delta(\chi, \theta)$$

where $\theta = f_0^{-1} f$. □

Example 17. *Following the discussion earlier in this section, the conjectures in section* 5 *are still equivalent to the statement in Conj.* 14 *applied to the case* $E = \mathbb{C}$ *with the isomorphism* $f = \lambda^{-1}$. *This gives* $R(\chi, \lambda^{-1}) = \delta(\chi, 1) = 1$, *and one obtains this intrinsic but essentially transcendent formulation of the conjecture due to Stark:*

For each $\alpha \in \mathrm{Aut}(\mathbb{C})$, we conjecture that

$$\frac{c(\chi^\alpha)}{c(\chi)^\alpha} = \delta(\chi^\alpha, \lambda \circ \lambda^{-\alpha})$$

5. Reduction to the abelian case and independence of S

We draw immediately from the previous section the following formulae concerning the numbers $A(\chi, f)$ introduced in §1.4 (or, more generally, in conjecture 14, suppose $E \subseteq \mathbb{C}$):

(1) $A(\chi + \chi', f) = A(\chi, f) \cdot A(\chi', f)$
(2) $A(\mathrm{Ind}\, \chi, f) = A(\chi, f)$
(3) $A(\mathrm{Infl}\, \chi, f) = A(\chi, f|_{\mathbb{C}X^H})$

This formalism allows one to reduce Stark's conjecture, on the one hand to the case $k = \mathbb{Q}$ (by passing to the Galois closure of K and induction), on the other hand to the case when characters are of dimension 1 (due to the theorem of Brauer, refer to Appendix F.)

Proposition 18. *We have*

(1) *If the conjecture is true for all finite Galois extensions* $K/\mathbb{Q}$, *then it is also true in general.*

(2) *If the conjecture is true for all irreducible characters of dimension* 1 *of all Galois extensions* K/k*, then it is also true in general.*

This being said, let us pass to the independence of the conjectures on the choice of S:
The set S fixed in section appears in the conjectures of section through an intermediary such as the L-function as well as the definition of the regulator. In fact, one has the

Proposition 19. *The truth of the conjecture 14 is independent of the choice of the set* S.

PROOF. We work with the version in . Suppose S is the initial set and let $S' = S \cup \{\mathfrak{p}\}$, where $\mathfrak{p}$ is a place of k not appearing in S. Denote by U', X', f',etc. the data in section with S replaced with S', as well as $c'(\chi), r'(\chi)$ the initial coefficient and the multiplicity of $L_{S'}(s, \chi)$ at $s = 0$ respectively. Finally, let $A'(\chi, f')$ be the resultant number as seen in section. We also assume that $f'\_\mathbb{C}X = f$. Let

$$B(\chi) = \frac{A'(\chi, f')}{A(\chi, f)}$$

We have to show that

Claim: $B(\chi)^\alpha = B(\chi^\alpha)$ for all $\alpha \in \mathrm{Aut}(\mathbb{C})$.

As in , and the formulae in , we note that it is sufficient to solve for $\chi(1) = 1$. This leads us to distinguish the two cases below. Let $\mathfrak{P}$ be a place of K lying above $\mathfrak{p}$ and $G_{\mathfrak{P}} \subseteq G$ its decomposition group.

Case-1: χ is not trivial on $G_{\mathfrak{P}}$

Then we have (as $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V^*$), we have $r(\chi) = r'(\chi); \mathrm{Hom}_G(V^*, \mathbb{C}X) = \mathrm{Hom}_G(V^*, \mathbb{C}X')$ and $R(\chi, f) = R'(\chi, f')$.

On the other hand, if χ is also not trivial on the inertia group $\mathrm{I}_{\mathfrak{P}}$ of $\mathfrak{P}$, we find that $L_S(s, \chi) = L_{S'}(s, \chi)$, and so $B(\chi) = 1 = B(\chi^\alpha)$, which implies the claim. Suppose to the contrary, $\chi(\mathrm{I}_{\mathfrak{P}}) = 1$, then $c'(\chi) = (1 - \chi(\sigma_{\mathfrak{P}}))c(\chi)$ and, as a consequence $B(\chi) = (1 - \chi(\sigma_{\mathfrak{P}}))^{-1}$, so that the claim is trivially true.

Case-2: $\chi(G_{\mathfrak{P}}) = 1$.

Due to property item 3 mentioned at the start of this section, it is enough to assume $G_{\mathfrak{P}} = 1$, which is to say that $\mathfrak{p}$ splits completely in the extension K/k. In this case, $L_{S'}(s,\chi) = (1 - N\mathfrak{p}^{-s})^{-1} L_S(s,\chi)$, so $c'(\chi) = c(\chi)\log N\mathfrak{p}$. On the other hand, $r'(\chi) = r(\chi) + 1$, and more precisely, if $\mathfrak{P}^h = \pi\mathcal{O}_K$, for $\pi \in K$:

$$\begin{cases} \mathbb{Q}U' & \simeq \mathbb{Q}U \oplus \mathbb{Q}[G]\cdot\pi \\ \mathbb{Q}X' & \simeq \mathbb{Q}X \oplus \mathbb{Q}[G]\cdot(\mathfrak{P} - \frac{1}{g}N_G w_0) \end{cases}$$

where w_0 is an arbitrary Archimedean place of K, $g = \#S$ and $N_G = \sum_{\sigma\in G}\sigma \in \mathbb{Q}[G]$.

In suitable bases, we obtain matrices for λ' and f' :

$$M(\lambda') = \begin{pmatrix} M(\lambda) & * \\ 0 & \log|\pi|_{\mathfrak{P}}\cdot 1_G \end{pmatrix}; M(f') = \begin{pmatrix} M(f) & * \\ 0 & 1_G \end{pmatrix}$$

As V is of dimension 1, it is easily deduced that the matrix corresponding to the endomorphism $(\lambda' \circ f')_V$ of $\mathrm{Hom}_G(V^*, \mathbb{C}X')$ can be put in the form :

$$M(\lambda') = \begin{pmatrix} M((\lambda\circ f)_V) & * \\ 0 & \log|\pi|_{\mathfrak{P}} \end{pmatrix}$$

where $\det M((\lambda\circ f)_V) = R(\chi, f)$.

Finally, one finds that $B(\chi) = \log|\pi|_{\mathfrak{P}}/\log N\mathfrak{p}$, a rational number which does not depend on χ. This concludes the proof of the proposition. □

6. Statement of Gross-Stark Conjecture

This section follows [Gro81][Ven14]

6.1. Gross's p-adic regulator. Recall that the definition of Stark regulator crucially depends on the logarithmic map λ defined in previous section. We also aim to find such a map. First, we shall build the the theory of p-adic absolute values.

Definition 20. *For each place $\mathfrak{P}$ of K, we can define the local absolute value $|\cdot|_{\mathfrak{P},\mathfrak{p}}$: $K_{\mathfrak{P}}^\times \to \mathbb{Z}_p^\times$ by*

$$|x|_{\mathfrak{P},\mathfrak{p}} = \begin{cases} 1 & \text{if } K_{\mathfrak{P}} \simeq \mathbb{C} \\ \mathrm{sign}(x) & \text{if } K_{\mathfrak{P}} \simeq \mathbb{R} \\ (\mathbb{N}\mathfrak{P})^{-v_{\mathfrak{P}}(x)} & \text{if } \mathfrak{P} \nmid \mathfrak{p} \\ (\mathbb{N}\mathfrak{P})^{-v_{\mathfrak{P}}(x)}\mathbb{N}_{K_{\mathfrak{P}}/\mathbb{Q}_p} & \text{if } \mathfrak{P} \mid \mathfrak{p} \end{cases}$$

Remark 21. (1) *It can be shown that the product formula holds for the local absolute values as well. More precisely,*

$$\prod_{\mathfrak{P}} |x|_{\mathfrak{P},\mathfrak{p}} = 1 \ \forall\ x \in K^\times$$

(2) *The local absolute values are not exactly the same as the usual absolute values. For example, if* x *is a totally positive unit, then* $|x|_{\mathfrak{P},p} = 1$ *for all places* $\mathfrak{P}$ *but usually if* $|x|_{\mathfrak{P}} = 1$ *for all places* $\mathfrak{P}$*, then* x *must be a root of unity.*

The second property is a useful property to have. So, we focus our attention to the subgroup

$$(K^\times)^- := \{x \in K^\times : |x|_{\mathfrak{P},p} = 1 \;\forall\, \mathfrak{P} \mid \infty\}$$

On this subgroup, we have the property that x *is a root of unity contained in* K *if and only if* $|x|_{\mathfrak{P}} = 1$ *for all finite places* $\mathfrak{P}$ *of* K*. [Gro81, Prop. 1.11]*
The above definition can be interpretated in the following manner as well. If $\tau \in G$ *is the complex conjugation, then*

$$(K^\times)^- = \{x \in K^\times : \tau(x) = -x\}$$

Next, fix the finite set S of places of K containing all the infinite places and the places dividing p. Let $U_{S,K}$ be the set of S-units of K and let $U^-_{S,K} = U_{S,K} \cap (K^\times)^\times$. Let $Y_{S,K}$ be the free abelian group on the set S and let $X_{S,K}$ be the subgroup of elements of degree 0 as in the previous section. Motivated from the logarithmic map $\lambda : U \to \mathbb{R}Y$, we define our local logarithmic map

$$\begin{aligned} \lambda_p : U_{S,K} &\to \mathbb{Q}_p Y_{S,K} \\ x &\mapsto \sum_{\mathfrak{P} \in S} \log_p |x|_{\mathfrak{P},p} \mathfrak{P} \end{aligned}$$

Due the product formula item 1, the image of λ_p lies in $\mathbb{Q}_p X_{S,K}$. We are interested in knowing whether the induced map $\lambda_p : \mathbb{Q}_p U^-_{S,K} \to \mathbb{Q}_p X_{S,K}$ is injective or not. The measure of how far the map is from being injective is quantified through the regulator. First, define

$$\begin{aligned} o_p : U^- &\to X^- \\ x &\mapsto \sum_{\mathfrak{P} \nmid \infty} f_{\mathfrak{P}} \nu_{\mathfrak{P}}(x) \mathfrak{P} \end{aligned}$$

Tensoring by $\mathbb{Q}_p$ over $\mathbb{Z}$ gives the induced map

$$o_p : \mathbb{Q}_p U^- \to \mathbb{Q}_p X^-$$

The map o_p is an isomorphism (just construct the inverse using the finiteness of the class number of K).

Definition 22. *We can define the Gross* p*-adic regulator via*

$$R_{p,K,S} = \det(\lambda_p \circ o_p^{-1} | \mathbb{Q}_p X^-)$$

6.2. Statement of the Gross-Stark conjecture. Let k be a totally real number field and $\overline{k}$ its algebraic closure. Let E be a field of characteristic 0 and V the finite dimensional vector space over E with an action of G_k. Consider the representation

$$\rho : G_k \to \mathrm{GL}(V)$$

that factors through the Galois group of a finite extension K/k. Such a representation is said to be totally odd if every complex multiplication acts as -1_V.

Fix a prime number p, and fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. This allows us to view χ as taking values in $\mathbb{C}$ or $\mathbb{C}_p$. Let S be a finite set of places of k containing all the infinite places of k. To the representation V, we have the S-depleted L-function

$$L_S(s, \rho) = \prod_{\mathfrak{p} \notin S} \det\left(1 - \sigma_{\mathfrak{p}} \mathbb{N}\mathfrak{p}^{-s} | V^{I_{\mathfrak{p}}}\right)^{-1} \tag{14}$$

Let S also contain all the divisors of p. Let

$$\omega : G(k(\mu_{2p})/k) \to (\mathbb{Z}/2p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$$

be the Teichmuller character. If $\alpha : E \to \mathbb{C}_p$ is an embedding, then V^α denotes the complex representation obtained by change of base. We have the following interpolation formula:

$$L_{p,S}(\omega^{1-n} \otimes V^\alpha, n) = a_S(V, n)^\alpha$$

where $a_S(V, n)$ is obtained via the relation

$$L_S(V^\beta, n) = a_S(V, n)^\beta$$

with $\beta : E \to \mathbb{C}$ an embedding.

The p-adic L function $L_p(\omega \otimes V^\alpha, s)$ is non-zero if and only if V is totally odd. Next, the Taylor expansion of $L_S(V^\beta, s)$ and $L_{p,S}(\omega \otimes V^\alpha, s)$ at $s = 0$ gives

- $L_S(V^\beta, s) \sim L(V^\beta)s^{r(V^\beta)}$
- $L_{p,S}(\omega \otimes V^\alpha, s) \sim L_p(V^\alpha)s^{r_p(V^\alpha)}$

Definition 23. *Define the regulators*

- $R(V^\beta) = \det\left(1 \otimes \lambda \circ f^{-1} | (V^\beta \otimes \mathbb{C}X^-)^G\right)$
- $R_p(V^\alpha) = \det\left(1 \otimes \lambda_p \circ o_p^{-1} | (V^\alpha \otimes \mathbb{C}X^-)^G\right)$

It can be shown that there is an algebraic number $A(V) \in E^\times$ such that for all embeddings $\beta : E \to \mathbb{C}$ both $r(V^\beta) = r(V)$ and $L(V^\beta) = R(V^\beta)A(V)^\beta$.

Conjecture 24. *For all embeddings* $\alpha : E \to \mathbb{C}_p$ *we have*

(1) $r_p(V^\alpha) = r(V)$

(2) $L_p(V^\alpha) = R_p(V^\alpha)A(V)^\alpha$

This conjecture can be reformulated as

Conjecture 25 (Gross-Stark conjecture)**.** *We have*

(1) $\mathrm{ord}_{s=0} L_{S,p}(\omega\chi, s) = r(\chi)$

(2) $\lim_{s\to 0} L_{S,p}(\omega\chi, s)/s^{r(\chi)} = \left((-1)^{r(\chi)} \prod_{\substack{\mathfrak{p}\in S \\ \chi(\mathfrak{p})=1}} f_{\mathfrak{p}} \right) \frac{h_K}{h_k} \frac{1}{|\mu(K)|Q} \prod_{\substack{\mathfrak{p}\in S \\ \chi(\mathfrak{p})\neq 1}} (1-\chi(\mathfrak{p}))$

CHAPTER 2

Cohomological interpretation of the conjecture

The notation from this chapter onwards follows [DDP11]. So, instead of K/k we deal with H/F defined below. We will further assume $\dim V = 1$.

Let F be a totally real field, and

$$\chi : G_F :\to \overline{\mathbb{Q}}^{\times}$$

be a totally odd character of the absolute Galois group of F. Let H be the cyclic extension of F cut out by $\mathrm{Ker}(\chi)$ (H is a CM extension as well; just look at the fixed field of the complex conjugation). χ can be seen as operating on the ideals of F via $\chi(\mathfrak{p}) = \chi(\mathrm{Frob}(\mathfrak{p}, H/F)) = 0$ if $\mathfrak{p}$ is ramified in H/F and $\chi(\mathfrak{p}) = \chi(\mathrm{Frob}(\mathfrak{p}, H/F))$ if $\mathfrak{p}$ is unramified in H/F.

Next, fix a prime number p, and embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. View χ as having values in $\mathbb{C}$ or $\mathbb{C}_p$. Let E be a finite extension of $\mathbb{Q}_p$ containing all values of χ.

In this section, we wish to reformulate $R_p(\chi)$ cohomologically.

For sake of completeness, we will restate the problem statement again.

Consider a finite set of places S of F containing all the infinite places. Then, the S-depleted L-function is defined to be

$$L_S(s, \chi) := \sum_{\substack{\gcd(\mathfrak{a}, v)=1 \\ \forall\, \mathfrak{p} \in S}} \chi(\mathfrak{a}) \mathbb{N}\mathfrak{a}^{-s} = \prod_{\mathfrak{p} \notin S} (1 - \chi(\mathfrak{p}) \mathbb{N}\mathfrak{p}^{-s})^{-1}$$

It is convergent for $\mathrm{Re}(s) > 1$ and has a holomorphic continuation to all of $s \in \mathbb{C}$ for nontrivial χ. Due to [DR80] we know of the existence of a continuous E-valued function

$$L_{S,p}(\chi\omega, -) : \mathbb{Z}_p \to \mathbb{C}_p$$

the p-adic L-function characterised by the interpolation property at negative integers $n \in \mathbb{Z}_{\leq 0}$:

$$L_{S,p}(\chi\omega, n) = L_S(\chi\omega^n, n)$$

A theorem of Siegel shows that $L_S(\chi, n)$ is algebraic and using the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ we can view the values to be p-adic. In fact, the function $L_{S,p}(\chi\omega, s)$ is meromophic on $\mathbb{Z}_p$ and regular as long as $\chi \neq \omega^{-1}$.

Let S_p be the set of places of F above p. We can take S to be $S_p \cup S_\infty \cup \{v : v \text{ ramified}\}$. We can partition S_p into

$$R = \{\mathfrak{p} \mid p : \chi(\mathfrak{p}) = 1\}, \quad R' = \{\mathfrak{p} \mid p : \chi(\mathfrak{p}) \neq 1\}$$

By the useful observation we made in theorem 10 we can deduce that $r := r_S(\chi) = \#R$. Gross conjectured that

Conjecture 26. $\mathrm{ord}_{s=0} L_{S,p}(\chi\omega, s) = r_S(\chi) = r$

Remark 27. *If* $T = S \backslash R$, *then*

$$L_S(\chi, s) = \left(\prod_{\mathfrak{p} \in R} 1 - \mathbb{N}\mathfrak{p}^{-s}\right) L_T(\chi, s)$$

Hence, $L_S(\chi, s) = 0$ *at* $s = 0$ *with order* $r_S(\chi)$. *By the interpolation property, the order of vanishing of* $L_{S,p}(\chi\omega, s)$ *at* $s = 0$ *is atleast* r.

The proof of this conjecture has been given in multiple papers, and is not the object of our thesis. What is interesting to us is the fact that Gross formulated a p-adic analog of Stark's conjecture as stated in conjecture 25 that gives an exact formula for the leading term of $L_{p,S}(\chi\omega, s)$ at $s = 0$. We will reconstruct the formula for our convenience.

Let X be the free abelian group generated by the primes in S_p and $U = \mathcal{O}_H[1/p]^\times$ be the group of p-units of H. X and U are naturally G-modules for the group $G = \mathrm{Gal}(H/F)$. We will consider the subspaces

$$U^- := \{u \in U : c(u) = u^{-1}\}, \quad X^- := \{x \in X : c(x) = -x\}$$

where c is complex conjugation.

Consider the two continuous homomorphisms for a prime $\mathfrak{P}$ in $\mathcal{O}_H$ lying above p:

$$\begin{aligned} \mathrm{ord}_{\mathfrak{P}} = o_{\mathfrak{P}} : & H_{\mathfrak{P}}^\times \to \mathbb{Z}, \\ \log_p \circ \mathbb{N}_{H_{\mathfrak{P}}/\mathbb{Q}_p} \ell_{\mathfrak{P}} = \ell_{\mathfrak{P}} : & H_{\mathfrak{P}}^\times \to \mathbb{Z}_p \end{aligned}$$

These homomorphisms induce two homomorphisms on the minus subspaces we had defined earlier:

$$\begin{aligned} o_p : & U^- \to X^- \\ & u \mapsto (o_{\mathfrak{P}}(u))_{\mathfrak{P} \in S_p} \\ \ell_p : & U^- \to X^- \otimes \mathbb{Z}_p \\ & u \mapsto (\ell_{\mathfrak{P}}(u))_{\mathfrak{P} \in S_p} \end{aligned}$$

The map o_p induces a $\mathbb{Q}[G]$-module isomorphism

$$U^- \otimes \mathbb{Q} \xrightarrow{\sim} X^- \otimes \mathbb{Q}$$

as we have seen earlier. If E is a finite extension of $\mathbb{Q}_p$ that contains the values of the character χ, we can define the χ^{-1} components of the minus subspaces U^- and X^-.

Let

$$U_\chi := (U^- \otimes E)^{\chi^{-1}} := \{u \in U \otimes E : \sigma u = \chi^{-1}(\sigma) u \,\forall\, \sigma \in G\}$$
$$X_\chi := (X^- \otimes E)^{\chi^{-1}} := \{x \in X \otimes E : \sigma u = \chi^{-1}(\sigma) u \,\forall\, \sigma \in G\}$$

The Galois equivariant form of Dirichlet's S-unit theorem tells us that U_χ is a finite dimensional E-vector space such that

$$\dim_E U_\chi = r_S(\chi) = r \tag{15}$$

The following conjecture is what we call the Gross-Stark conjecture:

Conjecture 28 (Gross-Stark)**.** *We have the equality*

$$\frac{L_{p,S}^{(r)}(\chi\omega, 0)}{r! L(\chi, 0)} = \mathcal{R}_p \prod_{\mathfrak{p} \in R'} (1 - \chi(\mathfrak{p}))$$

If we define the invariant

$$\mathcal{L}_{alg} = \frac{L_{p,S}^{(r)}(\chi\omega, 0)}{r! L(\chi, 0)} \cdot \frac{1}{\prod_{\mathfrak{p} \in R'} (1 - \chi(\mathfrak{p}))}$$

Then, the main result we want to prove is

THEOREM 29. *We have* $\mathcal{L}_{alg} = \mathcal{R}_p$

1. Rank-1 specific formulation

Definition 30. *Following Greenberg, the $\mathcal{L}$-invariant attached to χ is defined via the ratio*

$$\mathcal{L}(\chi) := -\frac{\ell_{\mathfrak{P}}(u_\chi)}{o_{\mathfrak{P}}(u_\chi)}$$

Remark 31.

- *The $\mathcal{L}$-invariant does not depend on the choice of the vector u_χ. Indeed, if u'_χ is another non-zero vector, then due to the 1-dimensionality of U_χ as a E-vector space we have $u'_\chi = \pi u_\chi$ with $\pi \in E^\times$. Thus, both the numerator and denominator have the extra factor π which cancels out.*
- *The $\mathcal{L}$-invariant is also independent of the choice of the prime $\mathfrak{P}$ above $\mathfrak{p}$. Indeed, if $\mathfrak{P}'$ were another prime, then using transitivity of G on $\mathfrak{P} \mid \mathfrak{p}$ we have $\mathfrak{P}' = \sigma\mathfrak{P}$ for some $\sigma \in G$. Consequently, $o_{\sigma(\mathfrak{P})} = o_{\mathfrak{P}}(\sigma^{-1} u_\chi) = o_{\mathfrak{P}}(\chi(\sigma) u_\chi) = \chi(\sigma) o_{\mathfrak{P}}(u_\chi)$, and $\ell_{\sigma\mathfrak{P}}(u_\chi) = \chi(\sigma)\ell_{\mathfrak{P}}(u_\chi)$ as well. Hence, the ratio is unaffected by the choice.*

We are now ready to state Gross's conjecture for our purposes.

Conjecture 32 (Gross)**.** *Let* F *be a totally real field,* H *a totally complex extension of* F*, and* $\chi : \mathrm{Gal}(H/K) \to \mathbb{C}^\times$ *a character of conductor* $\mathfrak{n}$*. If* $S = R \cup \{\mathfrak{p}\}$ *and* $r_S(\chi) = 1$*, then one can show that*

$$L'_{S,p}(\chi\omega, 0) = \mathcal{L}(\chi) L_R(\chi, 0)$$

To state the main theorem of DDP, we need to introduce some notation.

Definition 33.

$$\mathcal{L}_{an}(\chi, s) := \frac{-L_{S,p}(\chi\omega, 1-s)}{L_R(\chi, 0)}$$

$$\mathcal{L}_{an}(\chi) := \frac{L'_{S,p}(\chi\omega, 0)}{L_R(\chi, 0)} = \mathcal{L}'_{an}(\chi, 1)$$

This definition allows to rephrase the conjecture to asking whether $\mathcal{L}_{an}(\chi) = \mathcal{L}(\chi)$. The main theorem of DDP says that

Theorem 34. *Assuming that Leopoldt's conjecture holds for* F*, and the assumptions*

(1) *If* $|S_p| > 1$*, then the conjecture is true for all* χ*.*
(2) *If* $|S_p| = 1$ *and furthermore*

$$\mathrm{ord}_{k=1}(\mathcal{L}_{an}(\chi, k) + \mathcal{L}_{an}(\chi^{-1}, k)) = \mathrm{ord}_{k=1} \mathcal{L}_{an}(\chi^{-1}, k) \tag{16}$$

Then, the conjecture holds for both χ *and* χ^{-1}*.*

2. Cohomological interpretation

We define the cyclotomic character

$$\varepsilon_{cyc} : G_F \to \mathbb{Z}_p^\times$$

By $E(\chi)$ we will denote the vector space E on which G_F acts via the continuous action

$$\sigma \cdot x = \chi(\sigma) x$$

Similarly, $E(1)$ is equipped by the continuous action of the cyclotomic character. Thus, $E(\chi)(1)$ has a continuous action of $\chi\varepsilon_{cyc}$, and $E(\chi^{-1})$ the action via χ^{-1}.

3. Local Cohomology groups

Let ν be a place of F, $G_\nu \simeq G_{F_\nu}, I_\nu \subseteq G_\nu$ be choice of decomposition group and inertia group at ν.

Let $\mathfrak{p}_E = \langle \pi \rangle$ be the maximal ideal of $\mathcal{O}_E$. Tate's local duality gives a perfect pairing

$$\langle , \rangle_{\nu,n} : H^1(F_\nu, \mathcal{O}_E/\pi^n(\chi^{-1})) \times H^1(F_\nu, \mathcal{O}_E/\pi^n(\chi)(1)) \to \mathcal{O}_E/\pi^n$$

Taking limit $n \to \infty$ and then tensoring with E leads to the perfect pairing

$$\langle\, ,\, \rangle_v : H^1(F_v, E(\chi^{-1})) \times H^1(F_v, E(\chi)(1)) \to E \tag{17}$$

Definition 35. *If* M *is a* G_F*-module, then the inflation-restriction sequence gives*

$$0 \longrightarrow H^1(G_v/I_v, M^{I_v}) \xrightarrow{\text{Inf}} H^1(F_v, M) \xrightarrow{\text{res}_{I_v}} H^1(I_v, M)^{G_v/I_v}$$

The unramified classes classes are exactly the classes of $H^1(F_v, M)$ *that lie in the kernel of* res_{I_v}.

If $\chi(G_v) \neq 1$*, then* G_v/I_v *is a pro-cyclic group. Hence,*

$$\begin{aligned} H^1(G_v/I_v, \mathcal{O}_E/\pi^n(\chi^{-1})^{I_v}) &= \widehat{H}^{-1}(G_v/I_v, \mathcal{O}_E/\pi^n(\chi^{-1})^{I_v}) \\ &= (\mathcal{O}_E/\pi^n(\chi^{-1})^{I_v})/(\chi^{-1}(v) - 1) \end{aligned}$$

Thus, the quotient has bounded size independent of n. Or equivalently, if we take limit over n, then the limit has torsion. Consequently, tensoring with E tells us that $H^1(G_v/I_v, E(\chi^{-1})^{I_v}) = 0$ and hence there are no unramified classes.

Assume $\chi(G_v) = 1$. Then,

(1) $H^1(F_v, E(\chi^{-1})) = H^1(F_v, E) = \text{Hom}_{\text{cts}}(G_v, E)$ contains an unramified class

$$\kappa_{unr} : \text{Gal}(F_v^{unr}/F_v) \to \mathcal{O}_E, \quad \text{Frob}_v \mapsto 1$$

(2) If $v \mid p$, then we have a ramified class, namely the restriction of the logarithm of the cyclotomic character to G_v. In particular, we are concerned with

$$\kappa_{cyc} = \log_p(\varepsilon_{cyc}) \in H^1(F, E)$$

(3) Kummer theory gives a connecting homomorphism (which is an isomophism)

$$\delta_{v,n} : F_v^\times \otimes \mathbb{Z}/p^n\mathbb{Z} \to H^1(F_v, \mathbb{Z}/p^n\mathbb{Z}(1))$$

If we let $F_v^\times \widehat{\otimes} E := (\varprojlim_n F_v^\times \otimes \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}_p} E$, then the connecting homomorphism of Kummer theory becomes the isomorphism

$$\delta_v : F_v^\times \widehat{\otimes} E \to H^1(F_v, E(1))$$

(4) We can calculate the pairing. Let $u \in F_v^\times \widehat{\otimes} E$, note that
- $\langle \kappa_{unr}, \delta_v(u) \rangle_v = -\kappa_{unr}((u, \overline{F}_v|F_v)) = -\kappa_{unr}((u, F_v^{unr}|F_v)) = -o_v(u)$
- This uses some calculation as can be found in [AT90][Neu13][NSW08]

$$\begin{aligned} \langle \kappa_{cyc}, \delta_v(u) \rangle_v &= -(\log_p \circ \varepsilon_{cyc})((u, \overline{F}_v|F_v)) \\ &= -(\log_p \circ \varepsilon_{cyc})(\mathbb{N}_{F_v/\mathbb{Q}_p}(u), \overline{F}_v|\mathbb{Q}_p) \\ &= -\log_p(\mathbb{N}_{F_v/\mathbb{Q}_p}(u^{-1})) \\ &= -\ell_v(u) \end{aligned}$$

The above observation helps us view $\delta_v(F_v^\times \widehat{\otimes} E)$ as the orthogonal complement to κ_{unr} under the local Tate duality.

[DDP11, see Lemma 1.3] also calculate the dimensions of the two spaces $H^1(F_v, E(\chi)(1))$ and $H^1(F_v, E(\chi^{-1}))$. In fact, the dimensions of both the spaces are same, given by

$$\begin{cases} [F_v : \mathbb{Q}_p] & \chi(G_v) \neq 1, v \mid p \\ [F_v : \mathbb{Q}_p] + 1 & \chi(G_v) = 1, v \mid p \\ 1 & \chi(G_v = 1), v \nmid p\infty \\ 0 & \text{otherwise} \end{cases}$$

4. Global Cohomology groups

Recall the definition of unramified class

$$H^1_{unr}(F_v, E(\chi^{-1})) \simeq H^1(G_v/I_v, E(\chi^{-1})^{I_v})$$

The orthogonal complement of the space $H^1_{unr}(F_v, E(\chi^{-1}))$ under the local Tate duality is denoted by

$$H^1_{unr}(F_v, E(\chi)(1)) := \{u \in H^1(F_v, E(\chi)(1)) : \langle \kappa, u \rangle_v = 0 \ \forall \ \kappa \in H^1_{unr}(F_v, E(\chi^{-1}))\}$$

Under the observation

$$H^1_{unr}(F_v, E(\chi^{-1})) = \begin{cases} E \cdot \kappa_{unr} & \chi(G_v) = 1 \\ 0 & \text{otherwise} \end{cases}$$

we have

$$H^1_{unr}(F_v, E(\chi)(1)) = \begin{cases} \mathcal{O}_v^\times \widehat{\otimes} E & \chi(G_v) = 1 \\ H^1(F_v, E(\chi)(1)) & \text{otherwise} \end{cases}$$

Definition 36. *By $H^1_R(F, E(\chi^{-1}))$ we denote the subgroup of $H^1(F, E(\chi^{-1}))$ consisting of classes unramified outside of R and arbitrary at R.*

The corresponding orthogonal complements under the local Tate duality is denoted by $H^1_R(F, E(\chi)(1)) \subseteq H^1(F, E(\chi)(1))$.

Proposition 37. *The map*

$$\delta : U_\chi \to H^1_R(F, E(\chi)(1))$$

induced by Kummer theory is an isomorphism. In particular, as a E-vector space, $H^1_R(F, E(\chi)(1))$ has dimension r.

Proof. Consider a group G, with subgroup H. The five term inflation restriction exact sequence has the following (truncated) form:

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(H, A)^G \longrightarrow H^2(G/H, A^H)$$

If we set $G = \mathrm{Gal}(H/F)$, $G_H = \mathrm{Gal}(\overline{F}/H)$ and use the fact that $\mathrm{Gal}(\overline{F}/H) = \mathrm{Gal}(\overline{H}/H)$ as $\overline{H} = \overline{F}$, we get the following sequence:

$$0 \longrightarrow H^1(G, E(\chi)(1)^{G_H}) \xrightarrow{\text{inf}} H^1(F, E(\chi)(1)) \xrightarrow{\text{res}} H^1(H, E(\chi)(1))^G \longrightarrow H^2(G, E(\chi)(1)^{G_H})$$

The restriction map

$$\text{res} : H^1(F, E(\chi)(1)) \to H^1(H, E(\chi)(1))^G \tag{18}$$

is an isomorphism. Indeed, as $\varepsilon_{cyc}(G_H) \neq 1$, $E(\chi)(1)^{G_H} \subseteq E(1)^{G_H} = 0$ and hence $E(\chi)(1) = 0$. Therefore, the groups $H^1(G, E(\chi)(1)^{G_H})$ and $H^2(G, E(\chi)(1)^{G_H})$ are trivial, establishing the isomorphism.

Next, we claim that $H^1(F, E(\chi)(1))^G = H^1(F, E(1))^{\chi^{-1}}$. Indeed, suppose $[\sigma]$ is a cohomology class in $H^1(F, E(\chi)(1))^G$, then the classes $[g \cdot \sigma]$ and $[\sigma]$ are the same for any $g \in G$. But,

$$g \cdot \sigma(x) = g \cdot \sigma(gxg^{-1}) = \chi(g)\varepsilon_{cyc}(g)\sigma(gxg^{-1})$$

for all $g \in G, x \in H$. Then, the classes $[\chi^{-1}\sigma]$ and $[x \mapsto \varepsilon_{cyc}(g)\sigma(gxg^{-1})]$ are the same. We therefore have

$$H^1(F, E(\chi)(1)) \simeq H^1(H, E(\chi)(1))^G = H^1(H, E(1))^{\chi^{-1}} \simeq \left(H^\times \widehat{\otimes} E\right)^{\chi^{-1}}$$

where the last isomorphism is due to Kummer theory as shown in the previous section (replace F_v with H). Locally, we have the isomorphism:

$$H^1(F_v, E(\chi)(1)) \simeq \left(H_w^\times \widehat{\otimes} E\right)^{\chi^{-1}}$$

Next, consider the diagram:

$$\begin{tikzcd}
H^1(F, E(1)(\chi)) \arrow[r] \arrow[d] & H^1(H, E(1))^{\chi^{-1}} \arrow[r] \arrow[d] & \left(H^\times \widehat{\otimes} E\right)^{\chi^{-1}} \arrow[d] \\
\prod_{v \notin R} H^1(F_v, E(1)(\chi)) \arrow[r] & \prod_{w \notin R_H} H^1(H_w, E(1))^{\chi^{-1}} \arrow[r] & \prod_{w \notin R_H} \left(H_w^\times \widehat{\otimes} E\right)^{\chi^{-1}}
\end{tikzcd}$$

Recall that $H^1_{unr}(F_v, E(\chi)(1))$ is the orthogonal complement of $H^1_{unr}(F_v, E(\chi^{-1}))$ under the local Tate pairing. Such an element of $H^1(F_v, E(1)(\chi)) = \left(H_w^\times \widehat{\otimes} E\right)^{\chi^{-1}}$ has to be an unit at v by our calculations on the Tate pairing. Therefore, U_χ is precisely the pre-image of $H^1_R(F, E(\chi)(1))$. □

If W_v is a subspace of $H^1(F_v, E(\chi^{-1}))$, define

$$H^1_{W_v, v}(F, E(\chi^{-1})) \subseteq H^1_v(F, E(\chi^{-1}))$$

to be the subspace consisting of classes whose image under the map res_{I_p} lies in W_v. The dimension of this new subspace is also of interest to us. The following theorem addresses this question.

Proposition 38. *Suppose* $W = (W_v)_{v \in R}$ *is a family of subspaces,* $W_v \subseteq H^1(F_v, E)$ *is a subspace containing the unramified cocycle* $\kappa_{\mathrm{unr},v}$*. If we define* $H^1_{W,R}(F, E(\chi^{-1}))$ *to be the subspace of* $H^1_R(F, E(\chi^{-1}))$ *consisting of classes whose image under* res_{I_v} *lies in* W_v *for each* $v \in R$*. Then,*

$$\dim_E H^1_{W,R}(F, E(\chi^{-1})) = \left(\sum_{v \in R} \dim_E W_v\right) - |R|$$

Proof. The Poitou-Tate exact sequence from Galois cohomology gives the following exact sequence:

$$0 \to H^1_{[R]}(F, \mathcal{O}_E/\pi^n(\chi^{-1})) \to H^1_R(F, \mathcal{O}_E/\pi^n(\chi^{-1})) \to \prod_{v \in R} H^1(F_v, \mathcal{O}_E/\pi^n(\chi^{-1})) \to H^1_R(F, \mathcal{O}_E/\pi^n(\chi)(1))^\vee \tag{19}$$

where the first map is the inclusion as

$$H^1_{[R]}(F, \mathcal{O}_E/\pi^n(\chi^{-1})) := \{\sigma \in H^1_R(F, \mathcal{O}_E/\pi^n(\chi^{-1})) : \mathrm{res}_{I_v}\sigma = 0\}$$

and, the last map is the one induced from the local Tate pairing. Consider the inflation-restriction sequence:

$$0 \longrightarrow H^1(\mathrm{Gal}(H/F), \mathcal{O}_E/\pi^n(\chi^{-1})) \xrightarrow{\mathrm{inf}} H^1(G_F, \mathcal{O}_E/\pi^n(\chi^{-1})) \xrightarrow{\mathrm{res}} H^1(G_F, \mathcal{O}_E/\pi^n(\chi^{-1}))$$

Now, $H^1(\mathrm{Gal}(H/F), \mathcal{O}_E/\pi^n(\chi^{-1})) = {}_N\mathcal{O}_E/\pi^n/I_{\mathrm{Gal}(H/F)}\mathcal{O}_E/\pi^n$ as $\mathrm{Gal}(H/F)$ is a finite cyclic group. The right hand side is finite group of size bounded independently of n. This means the classes of $H^1_{[R]}(F, \mathcal{O}_E/\pi^n(\chi^{-1}))$ map to (with kernel bounded independently of n) group homomorphisms from $G_H \to \mathcal{O}_E/\pi^n$ that are everywhere unramified. Such maps factor through class group $\mathrm{Cl}(H)$ of H. Hence, $H^1_{[R]}(F, E(\chi^{-1}))$ has bounded cardinality as n goes to infinity. Taking the limit of $n \to \infty$ and tensoring with E gives the sequence:

$$0 \to H^1_R(F, E(\chi^{-1})) \to \prod_{v \in R} H^1(F_v, E(\chi^{-1})) \to H^1_R(F, E(\chi)(1))^\vee \tag{20}$$

The element $\kappa_{\mathrm{unr},v}$ is mapped to a non-zero element of $H^1_R(F, E(\chi)(1))^\vee$ as the Tate pairing is non-degenerate. We want to show that the images of κ_{unr}, v are linearly independent. This follows from the fact that $(\mathcal{O}_H[1/p]^\times \otimes E)^{\chi^{-1}}$ is 1-dimensional as an E-vector space. As $H^1_R(F, E(\chi)(1))^\vee$ is r-dimensional by proposition 37, the last arrow in eq. (20) is surjective. In fact, the sequence restricts to

$$0 \to H^1_R(F, E(\chi^{-1})) \to \prod_{v \in R} W_v \to H^1_R(F, E(\chi)(1))^\vee \tag{21}$$

for subspaces $W_v \subseteq H^1(F_v, E)$ containing $\kappa_{unr,v}$. This concludes the proof. □

We will end this section by recording an important proposition that showcases the major difference between the rank one and the general case.

THEOREM 39. *The restriction map*

$$\prod_{i=1}^{r} \mathrm{res}_{I_{\mathfrak{p}_i}} : H^1_R(G_F, E(\chi^{-1})) \to \prod_{i=1}^{r} H^1(I_{\mathfrak{p}_i}, E)$$

is injective.

5. $\mathcal{L}$-invariant and proof strategy

We have proven that $\dim_E U_\chi = r$. Let $u_1, \dots, u_r$ be an E-basis for U_χ. Write $R = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. For each $\mathfrak{p}_i$, we have the continuous homomorphisms:

$$\mathrm{ord}_{\mathfrak{p}_i} = o_i : F_{\mathfrak{p}_i}^\times \to \mathbb{Z},$$
$$\log_p \circ \mathbb{N}_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} = \ell_i : F_{\mathfrak{p}_i}^\times \to \mathbb{Z}_p$$

For each $\mathfrak{p}_i$, choose $\mathfrak{P}_i$ of H lying above $\mathfrak{p}_i$. Then, using

$$\mathcal{O}_H[1/p] \subseteq H \subseteq H_{\mathfrak{P}_i} \simeq F_{\mathfrak{p}_i}$$

we can evaluate o_i, ℓ_i on elements of $\mathcal{O}_E[1/p]^\times$ and extend linearly to E and obtains maps:

$$o_i, \ell_i : \mathcal{O}_H[1/p]^\times \otimes E \to E$$

Gross's p-adic regulator can be explicitly written as the ratio:

$$\mathcal{R}_p(\chi) = \frac{\det(-\ell_i(u_j))}{\det(o_i(u_j))}$$

where (i, j) run from $(1, 1)$ to (r, r).

Remark 40. *The ratio is well-defined. As the χ^{-1} component of the group of $\mathfrak{p}_i$ units of H is 1-dimensional for each i, we can choose our u_js such that $o_i(u_j) = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta. The ratio does not depend on the choice of u_js as well. For two units differ by an unit, the ℓ_i maps the two units to the same quantity and so does the order. The ratio is also independent of the choice of $\mathfrak{P}_i$. Indeed, if we replace $\mathfrak{P}_i$ with $\sigma\mathfrak{P}_i$, then both the numerator and denominator are scaled by $\chi(\sigma)$ rendering the ratio same as before.*

Define the subspace of **cyclotomic classes**

$$H^1_{cyc}(\chi) \subseteq H^1_R(G_F, E(\chi^{-1}))$$

to consist of those classes κ whose restriction $\mathrm{res}_{\mathfrak{p}_i} \kappa \in H^1(G_{\mathfrak{p}_i}, E)$ lies in the span of $\kappa_{unr,\mathfrak{p}_i}$ and $\kappa_{cyc,\mathfrak{p}_i}$. From proposition 38, it is clear that $\dim_E H^1_{cyc}(\chi) = r$. Let $\kappa_1, \dots, \kappa_r$ be an E-basis for $H^1_{cyc}(\chi)$, and for each $\mathfrak{p}_j \in R$, let

$$\mathrm{res}_{\mathfrak{p}_j} \kappa_i = x_{ij}\kappa_{unr,\mathfrak{p}_j} + y_{ij}\kappa_{cyc,\mathfrak{p}_j}$$

Again, inspired by Greenberg, we define the algebraic invariant to be

$$\mathcal{L}_{alg} = \frac{\det(x_{ij})}{\det(y_{ij})}$$

If $\det(y_{ij}) = 0$, then we can find a basis $\kappa_1', \dots, \kappa_r'$ such that

$$\mathrm{res}_{\mathfrak{p}_j} \kappa_i' = x_{ij}\kappa_{unr,\mathfrak{p}_r}$$

This means the two spaces $H^1_{cyc}(\chi), H^1_W(G_F, E(\chi^{-1}))$ are equal where $W = (W_i)_{i=1,\dots,r}$ is a family of subspaces such that $W_i = \mathrm{span}\{\kappa_{unr,\mathfrak{p}_i}, \kappa_{cyc,\mathfrak{p}_i}\}$ for $i = 1, \dots, r-1$ and $W_r = \mathrm{span}\{\kappa_{unr,\mathfrak{p}_r}\}$ But clearly, the dimension of the two spaces are different. A contradiction. Hence, $\det(y_{ij})$ must be non-zero and thus the ratio is well-defined.

Proposition 41. *Let $\kappa \in H^1_R(G_F, E(\chi^{-1}))$ and $u \in U_\chi$. We can show that*

$$\sum_{i=1}^{r} (\mathrm{res}_{\mathfrak{p}_i} \kappa)(u) = 0 \tag{22}$$

Proof. From the Poitou-Tate duality, the product of the images of $H^1_R(G_F, E(\chi^{-1}))$ and $H^1_R(G_F, E(\chi)(1))$ under the product of restriction maps $\mathrm{res}_{\mathfrak{p}_i}$ are orthogonal under the local Tate duality map:

$$\langle\, , \,\rangle_R : \prod_{i=1}^{r} H^1(G_{\mathfrak{p}_i}, E(\chi^{-1})) \times \prod_{i=1}^{r} H^1(G_{\mathfrak{p}_i}, E(\chi)(1)) \xrightarrow{\sum\langle , \rangle_{\mathfrak{p}_i}} E$$

Using the isomorphism $\delta : U_\chi \simeq H^1_R(G_F, E(\chi)(1))$ we have

$$\begin{aligned} 0 = \langle \kappa, \delta(u)\rangle_R &= \sum_{i=1}^{r} \langle \mathrm{res}_{\mathfrak{p}_i}(\kappa), \delta(u)\rangle_{\mathfrak{p}_i} \\ &= \sum_{i=1}^{r} (\mathrm{res}_{\mathfrak{p}_i}(\kappa))(u) \end{aligned}$$

□

Corollary 42. *We have*

$$\mathcal{L}_{alg} = \mathcal{R}_p(\chi)$$

Proof. Let $u_1, \dots, u_r$ be a basis of U_χ as an E-vector space and $\kappa_1, \dots, \kappa_r$ a basis for $H^1_{cyc}(\chi)$. Using the previous proposition, we have

$$\sum_{i=1}^{r} (\mathrm{res}_{\mathfrak{p}_i} \kappa_j)(u_k) = 0$$

for all $j, k = 1, \ldots, r$. As a matrix this means

$$\begin{pmatrix} x_{11} & x_{21} & \cdots & x_{r1} \\ x_{12} & x_{22} & \cdots & x_{r2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1r} & x_{2r} & \cdots & x_{rr} \end{pmatrix} \begin{pmatrix} o_1(u_1) & o_1(u_2) & \cdots & o_1(u_r) \\ o_2(u_2) & o_2(u_2) & \cdots & o_2(u_r) \\ \vdots & \vdots & \vdots & \vdots \\ o_r(u_1) & o_r(u_2) & \cdots & o_r(u_r) \end{pmatrix}$$

$$= - \begin{pmatrix} y_{11} & y_{21} & \cdots & y_{r1} \\ y_{12} & y_{22} & \cdots & y_{r2} \\ \vdots & \vdots & \vdots & \vdots \\ y_{1r} & y_{2r} & \cdots & y_{rr} \end{pmatrix} \begin{pmatrix} \ell_1(u_1) & \ell_1(u_2) & \cdots & \ell_1(u_r) \\ \ell_2(u_2) & \ell_2(u_2) & \cdots & \ell_2(u_r) \\ \vdots & \vdots & \vdots & \vdots \\ \ell_r(u_1) & \ell_r(u_2) & \cdots & \ell_r(u_r) \end{pmatrix}$$

Hence,

$$\mathcal{L}_{alg} = \frac{\det(y_{ij})}{\det(x_{ij})} = \frac{\det(\ell_i(u_k))}{\det(o_i(u_k))}$$

□

6. Formula for $\mathcal{L}$ invariant in rank-1 and proof strategy

Definition 43. *If W_{cyc} is the subspace of $H^1(F_\mathfrak{p}, E)$ spanned by the classes κ_{unr} and κ_{cyc}, define*

$$H^1_{\mathfrak{p},cyc}(F, E(\chi^{-1})) := H^1_{\mathfrak{p},W_{cyc}}(F, E(\chi^{-1})) \tag{23}$$

By the previous proposition, the space $H^1_{\mathfrak{p},cyc}(F, E(\chi^{-1}))$ is 1-dimensional over E. Thus, any non-trivial element κ in this space is of the form

$$\mathrm{res}_{I_\mathfrak{p}}(\kappa) = x\kappa_{unr} + y\kappa_{cyc}$$

for some $x, y \in E$. $y \neq 0$ for it contradicts the dimension when W_{cyc} is spanned by just κ_{unr}. As the space is 1-dimensional, the choice of κ does not change the ratio y/x.

By the reciprocity law of Global Class Field theory, we have

$$\begin{aligned} \langle \kappa, \delta(u_\chi) \rangle &= \sum_v \langle \mathrm{res}_{I_v} \kappa, \delta_v(u_\chi) \rangle_v \\ &= \langle \mathrm{res}_{I_\mathfrak{p}} \kappa, \delta_\mathfrak{p}(u_\chi) \rangle_\mathfrak{p} \\ &= x \langle \kappa_{unr}, \delta_\mathfrak{p}(u_\chi) \rangle_\mathfrak{p} + y \langle \kappa_{cyc}, \delta_\mathfrak{p}(u_\chi) \rangle_\mathfrak{p} \\ &= -x \cdot o_\mathfrak{p}(u_\chi) + y \ell_\mathfrak{p}(u_\chi) \end{aligned}$$

But $\langle \kappa, \delta(u_\chi) \rangle = 0$ by definition. Hence, $\mathcal{L}(\chi) = -x/y$.

Conjecture 44. *The above observation allows us to reduce our theorem to the following:*

There exists a nontrivial class $\kappa \in H^1_{\mathfrak{p},\mathrm{cyc}}(F, E(\chi^{-1}))$ *satisfying*

$$\mathrm{res}_{I_{\mathfrak{p}}}(\kappa) = x\kappa_{\mathrm{unr}} + y\kappa_{\mathrm{cyc}}$$

CHAPTER 3

Construction of cusp form

Main references are [DDP11][Shi78][Gar90]

1. Hilbert Modular Forms

Let F be a totally real number field of degree $n = [F : \mathbb{Q}]$. The embeddings be $\tau_1, \dots, \tau_n$. If $a \in \mathcal{O}_F$, then a can be seen as an element of $F \hookrightarrow \mathbb{R}$ via the tuple $a = (a_i := \tau_i a)_i$.

Let ψ be a narrow ray class character modulo $\mathfrak{b}$ with sign $r \in \mathbb{F}_2^n$. If $\alpha \in \mathcal{O}_F$ is relatively prime to $\mathfrak{b}$, we can define a character associated to ψ by

$$\psi_f : (\mathcal{O}_F/\mathfrak{b})^\times \to \overline{\mathbb{Q}}^\times, \quad \alpha \mapsto \operatorname{sign}(\alpha)^r \psi(\langle \alpha \rangle)$$

Fix an integer k. Let $\lambda \in \mathrm{Cl}^+(F)$ be an ideal class, choose a representative fractional ideal $\mathfrak{t}_\lambda$. Let $\mathrm{GL}_2^+(F)$ denote the 2×2 matrices with elements from F such that the determinant is totally positive (all galois conjugates are positive). Define the level

$$\Gamma_\lambda := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) : a, d \in \mathcal{O}_F, b \in \mathfrak{t}_\lambda^{-1}\mathfrak{d}^{-1}, c \in \mathfrak{b}\mathfrak{t}_\lambda\mathfrak{d}, ad - bc \in \mathcal{O}_F^\times \right\}$$

The space $M_k(\mathfrak{b}, \psi)$ of Hilbert modular forms of level $\mathfrak{b}$ and character ψ consists of functions $f = (f_\lambda)_{\lambda \in \mathrm{Cl}^+(F)}$ with

$$f_\lambda : \mathcal{H}^n \to \mathbb{C}$$

such that each function f_λ satisfies

$$f_\lambda|_\gamma = \psi_f(a) f_\lambda$$

for all $\gamma \in \Gamma_\lambda$ where the slash operator $|_\gamma$ is defined to be

$$f_\lambda|_\gamma(z) := \det(\gamma)^{k/2}(cz + d)^{-k} f_\lambda(\gamma z)$$

$$(cz + d)^k := \prod_{i=1}^{n} (c_i z_i + d_i)^k$$

$$\det(\gamma)^{k/2} := \prod_{i=1}^{n} \det(\gamma_i)^{k/2}$$

$$\gamma z := \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \cdots, \frac{a_n z_n + b_n}{c_n z_n + d_n} \right)$$

The function $f \in M_k(\mathfrak{b}, \psi)$ must also satisfy

$$S(\mathfrak{m})f = \psi(\mathfrak{m})f \ \forall \ \gcd(\mathfrak{m}, \mathfrak{b}) = 1 \tag{24}$$

It can be shown that f_λ has a Fourier expansion

$$f_\lambda(z) = a_\lambda(0) + \sum_{\substack{b \in t_\lambda \\ b >> 0}} a_\lambda(b) \exp(2\pi i \mathrm{Tr}_{F/\mathbb{Q}}(x)) \tag{25}$$

Definition 45. *The coefficients $a_\lambda(b)$ are called unnormalised Fourier coefficients of f. We define the normalised Fourier coefficients $c(\mathfrak{m}, f), c(0, f)$ to be*

$$c(\mathfrak{m}, f) := a_\lambda(b)\mathbb{N}t_\lambda^{-k/2}, \ \ c(0, f) := a_\lambda(0)\mathbb{N}t_\lambda^{-k/2}$$

where an integral ideal $\mathfrak{m} = bt_\lambda^{-1}$ for a totally positive element b and an unique λ.

Remark 46. *Note that the definition does not depend on the choice of b. Indeed, any other choice of b would differ by a totally positive unit ϵ, and modularity condition would imply $f_\lambda(\epsilon z)\mathbb{N}\epsilon^{k/2} = f_\lambda(z)$.*

Definition 47. *If for each $\gamma \in \mathrm{GL}_2(F)^+$ and $\lambda \in \mathrm{Cl}^+(F)$, the function $f|_\gamma$ has constant term 0, then we say f is a cusp form. The space of cusp forms of weight k, level $\mathfrak{b}$ and character ψ is denoted by $S_k(\mathfrak{b}, \psi)$.*

2. Eisenstein series

A standard example of Hilbert modular forms come from Eisenstein series associated to two narrow ray class characters.

Let $\mathfrak{a}, \mathfrak{b}$ be two integral ideals of F, η, ψ be two narrow ray class characters modulo $\mathfrak{a}, \mathfrak{b}$ respectively. Also, suppose the signs of η, ψ are q, r satisfying

$$q + r \equiv (k, k, \ldots, k) \mod 2\mathbb{Z}^n$$

Then it can be shown that [DDP11, Proposition 2.1][Shi78, Proposition 3.4] there exists $E_k(\eta, \psi) \in M_k(\mathfrak{a}\mathfrak{b}, \eta\psi)$ such that

$$c(\mathfrak{m}, E_k(\eta, \psi)) = \sum_{\mathfrak{r} | \mathfrak{m}} \eta(\mathfrak{m}/\mathfrak{r})\psi(\mathfrak{r})\mathbb{N}\mathfrak{r}^{k-1} \tag{26}$$

In fact, the constant term of the Eisenstein series can be computed explicitly[DDP11, Proposition 2.1], as seen in the following

(27)
$$c_\lambda(0, E_k(\eta,\psi)) = \begin{cases} 2^{-n}\eta^{-1}(t_\lambda)L_S(\psi\eta^{-1}, 1-k) & k>1, \mathfrak{a}=1, \\ 0 & k>1, \mathfrak{a}\neq 1, \\ 2^{-n}\eta^{-1}(t_\lambda)L_S(\psi\eta^{-1}, 0) & k=1, \mathfrak{a}=1, \mathfrak{b}\neq 1, \\ 2^{-n}\psi^{-1}(t_\lambda)L_S(\eta\psi^{-1}, 0) & k=1, \mathfrak{a}\neq 1, \mathfrak{b}=1, \\ 2^{-n}(\eta^{-1}(t_\lambda)L_S(\psi\eta^{-1}, 0)+\psi^{-1}(t_\lambda)L_S(\eta\psi^{-1}, 0)) & k=1, \mathfrak{a}=1, \mathfrak{b}=1, \\ 0 & k=1, \mathfrak{a}, \mathfrak{b}\neq 1 \end{cases}$$

3. Construction of cusp form in $r = 1$

Definition 48. *Whenever* $L(\psi, 1-k) \neq 0$*, the normalised Eisenstein series can be defined as*

$$G_k(1,\psi) := \frac{2^n}{L(\psi, 1-k)} E_k(1,\psi) \tag{28}$$

Using the values of $E_k(1,\psi)$ as in the last proposition, we observe that $c_\lambda(G_k(1,\psi), 0) = 1$.

Recall that $\chi : G_F \to \overline{\mathbb{Q}}^\times$ is a character of conductor $\mathfrak{n}$ and $\chi(\mathfrak{p}) = 1$. Let

$$\mathfrak{n}_R = \operatorname{lcm}\left(\mathfrak{n}, \prod_{\mathfrak{q}|p, \mathfrak{q}\neq\mathfrak{p}} \mathfrak{q}\right), \quad \mathfrak{n}_S = \operatorname{lcm}\left(\mathfrak{n}, \mathfrak{p}\prod_{\mathfrak{q}|p, \mathfrak{q}\neq\mathfrak{p}} \mathfrak{q}\right)$$

We will view χ as a character χ_R(resp. $\chi_S = \chi\omega^{1-k}$) with modulus $\mathfrak{n}_R$(resp. $\mathfrak{n}_S$).

We will concern ourselves with the modular form

$$P_k := E_1(1, \chi_R)G_{k-1}(1, \omega^{1-k}) \in M_k(\mathfrak{n}_S, \chi\omega^{1-k}) \tag{29}$$

The modular form $G_{k-1}(1,\omega^{1-k})$ makes sense as can be seen from the functional equation of $L_S(\chi, s)$.

[Wil86]Every modular form in $M_k(\mathfrak{n}_S, \chi\omega^{1-k})$ can be written uniquely as a linear combination of a cusp form and the Eisenstein series $E_k(\eta,\psi)$ with the pair (η,ψ) running over the set J of characters with modulus $\mathfrak{m}_\eta, \mathfrak{m}_\psi$ respectively satisfying

$$\mathfrak{m}_\eta\mathfrak{m}_\psi = \mathfrak{n}_S, \quad \eta\psi = \chi\omega^{1-k} \tag{30}$$

More concretely,

$$P_k = (\text{cusp form}) + \sum_{(\eta,\psi)\in J} a_k(\eta,\psi)E_k(\eta,\psi) \tag{31}$$

As we are interested in constructing a cusp form, we would like to remove the contribution from Eisenstein series in the above expression. This is achieved with the help of an appropriate Hecke operator as will be developed later in this section. We are interested in the coefficients $a_k(\chi, \omega^{1-k})$ and $a_k(1, \chi\omega^{1-k})$ for it turns out that their values are ratios of the classical L-functions and p-adic L-functions. We will record this fact in the following

Proposition 49. *[DDP11, Proposition 2.6, 2.7] If $k \in \mathbb{Z}_{\geq 2}$, then*

$$a_k(1, \chi\omega^{1-k}) = \frac{L_R(\chi, 0)}{L_{S,p}(\chi\omega, 1-k)} = -\mathcal{L}_{an}(\chi, k)^{-1}$$

If $k \in \mathbb{Z}_{>2}$ and $\mathfrak{p}$ is the unique prime above p ($|S_p| = 1$), then

$$a_k(\chi, \omega^{1-k}) = \frac{L_R(\chi^{-1}, 0)}{L_{S,p}(\chi^{-1}, \omega, 1-k)} \langle \mathbb{N}\mathfrak{n} \rangle^{k-1} = -\mathcal{L}_{an}(\chi^{-1}, k)^{-1} \langle \mathbb{N}\mathfrak{n} \rangle^{k-1}$$

PROOF. It follows simply by comparing coefficients on both sides of the equation

$$P_k = (\text{cusp form}) + \sum_{(\eta,\psi)\in J} a_k(\eta, \psi) E_k(\eta, \psi)$$

and using the linear independence of characters of the narrow ray class group $\mathrm{Cl}^+(F)$. □

If $\mathfrak{q}$ is a prime ideal, we denote by $T_\mathfrak{q}, U_\mathfrak{q}$ the Hecke operators. They act on the Eisenstein series in the following manner

$$\begin{aligned}
T_\mathfrak{q} E_k(\eta, \psi) &= (\eta(\mathfrak{q}) + \psi(\mathfrak{q})(\mathbb{N}\mathfrak{q})^{k-1}) E_k(\eta, \psi) && \mathfrak{q} \nmid \mathfrak{n}_S \\
U_\mathfrak{q} E_k(\eta, \psi) &= (\eta(\mathfrak{q}) + \psi(\mathfrak{q})(\mathbb{N}\mathfrak{q})^{k-1}) E_k(\eta, \psi) && \mathfrak{q} \mid \mathfrak{n}_S \\
&= \eta(\mathfrak{q}) E_k(\eta, \psi) && \mathfrak{q} \nmid \mathfrak{m}_\eta \\
&= \psi(\mathfrak{q})(\mathbb{N}\mathfrak{q})^{k-1} E_k(\eta, \psi) && \mathfrak{q} \mid \mathfrak{m}_\eta
\end{aligned}$$

Definition 50. *Remember that $E/\mathbb{Q}_p$ is a finite extension containing the values of χ. Consider the $\mathcal{O}_E$-submodule $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_E) \subseteq M_k(\mathfrak{n}_S, \chi\omega^{1-k})$ consisting of modular forms with the normalised Fourier coefficients lying in the ring $\mathcal{O}_E$. The ordinary projector or Hida's idempotent [Hid93] defined as*

$$e := \lim_{r\to\infty} \left(\prod_{\mathfrak{q}|p} U_\mathfrak{q} \right)^{r!} \tag{32}$$

is an idempotent in $\mathrm{End}(M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_E))$*. We can extend it to* $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; E)$ E*-linearly using the fact that*

$$M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_E) \otimes_{\mathcal{O}_E} E = M_k(\mathfrak{n}_S, \chi\omega^{1-k}; E)$$

It is easy to see that $eE_k(\eta, \psi) = E_k(\eta, \psi)$ if $\gcd(p, \mathfrak{m}_\eta) = 1$ and 0 otherwise. This allows us to formulate

Proposition 51. *[DDP11, Proposition 2.8] If $P_k^0 = eP_k$, then*

$$P^0 = (\text{ an ordinary cusp form}) + \sum_{(\eta,\psi)\in J^0} a_k(\eta,\psi)E_k(\eta,\psi)$$

where (η,ψ) runs through the set J_0 consisting of the pairs (η,ψ) such that

$$m_\eta m_\psi = \mathfrak{n}_S, \quad \eta\psi = \chi\omega^{1-k}, \gcd(p, m_\eta) = 1 \tag{33}$$

Lemma 52. *[DDP11, Lemma 2.9]*

1. *For each $(\eta,\psi) \in J^0$ with $\eta \notin \{1,\chi\}$, we have a Hecke operator $T_{(\eta,\psi)}$ such that*
$$T_{(\eta,\psi)}E_k(\eta,\psi) = 0, \quad T_{(\eta,\psi)}E_1(1,\chi_S) = 1$$
2. *If the set $R = S - \{\mathfrak{p}\}$ contains a prime above p, then there is a Hecke operator $T_{(\chi,\omega^{1-k})}$ satisfying*
$$T_{(\chi,\omega^{1-k})}E_k(\chi,\omega^{1-k}) = 0, \quad T_{(\chi,\omega^{1-k})}E_1(1,\chi_S) = 1$$

If F has prime above p other than $\mathfrak{p}$, then set

$$u_k := \frac{1}{1+\mathcal{L}_{an}(\chi,k)}, \quad w_k := \frac{\mathcal{L}_{an}(\chi,k)}{1+\mathcal{L}_{an}(\chi,k)}, \quad v_k := 0$$

If $\mathfrak{p}$ is the unique prime in F above p, then set

$$u_k := \frac{\mathcal{L}_{an}(\chi,k)^{-1}}{c_k}, \quad w_k := \frac{1}{c_k}, \quad v_k := \frac{\mathcal{L}_{an}(\chi^{-1},k)^{-1}\langle \mathbb{N}\mathfrak{n}\rangle^{k-1}}{c_k}$$

where

$$c_k = \mathcal{L}_{an}(\chi,k)^{-1} + \mathcal{L}_{an}(\chi^{-1},k)^{-1}\langle \mathbb{N}\mathfrak{n}\rangle^{k-1} + 1$$

As a direct corollary to the lemma and the notations above, we have

Theorem 53. *[DDP11, Corollary 2.10] If $H_k := u_kE_k(1,\chi\omega^{1-k}) + v_kE_k(\chi,\omega^{1-k}) + w_kP_k^0$, then the modular form*

$$F_k := \left(\prod_{(\eta,\psi)} T_{(\eta,\psi)}\right) H_k$$

is a cusp form belonging to $S_k(\mathfrak{n}_S, \chi\omega^{1-k})$. The product is over J^0 with $\eta \neq 1$ if F has primes other than $\mathfrak{p}$ above p and the product is over J_0 with $\eta \neq 1,\chi$ if $\mathfrak{p}$ is the only prime in F above p.

4. General rank, i.e. $r \geq 1$

Please come back to this section after reading the next chapter, section 1.

Recall that we have constructed a Hida family $\mathcal{G} \in \mathcal{M}^o(1,\omega^{-1}) \otimes_\Lambda \Lambda_{(0)}$ with the property that $v_0(\mathcal{G}) = 1$ and $c_\lambda(0,\mathcal{G}) = 1$ for all $\lambda \in \mathrm{Cl}^+(F)$.

We set

$$G_k := \nu_k(\mathcal{G}) \in M_k(p, \omega^{-k})$$

For an integer $k \geq 1$, we define (Case 1: When R' is non-empty)

$$H_k := E_k(1, \chi\omega^{1-k}) - E_1(1, \chi_{R'})G_{k-1}\frac{L_p(\chi\omega, 1-k)}{L(\chi_{R'}, 0)} \tag{34}$$

Here $L(\chi_{R'}, 0)$ is the R'-depleted L function and hence upto multiplication by $2^{-[F:\mathbb{Q}]}$, it is the value of the constant term of $E_1(1, \chi_{R'})$.

When $R' = \emptyset$, we let

$$H_k := E_k(1, \chi\omega^{1-k}) - E_1(1, \chi)G_{k-1}\frac{L_p(\chi\omega, 1-k)}{L(\chi, 0)} + E_k(\chi, \omega^{1-k})\frac{L_p(\chi\omega, 1-k)}{L(\chi, 0)}\frac{L(\chi^{-1}, 0)}{L_p(\chi^{-1}\omega, 1-k)} \tag{35}$$

In eq. (34), as per our observations above and eq. (27) the constant term is 0. Similarly, in eq. (35) as well, the constant term is 0.

CHAPTER 4

Hida Families and Hecke Algebras

1. Λ-adic Eisenstein series

Recall that the Iwasawa algebra $\Lambda \simeq \mathcal{O}_E[[T]]$ is topologically generated over $\mathcal{O}_E$ by the functions of the form $k \mapsto u^k$ with $u \in 1 + 2p\mathbb{Z}_p$. For each $k \in \mathbb{Z}_p$ we have a homomorphism

$$\nu_k : \Lambda \to \mathcal{O}_E, T \mapsto u^{k-1} - 1$$

called the specialisation to weight k. $\Lambda_{(k)}$ will denote the localisation of Λ at $\mathrm{Ker}\nu_1$, and sometimes we will view ν_k as a homomorphism from $\Lambda_{(1)} \to E$.

Definition 54. *A family $\mathcal{F} = \{c(\mathfrak{m}, \mathcal{F}), c_\lambda(0, \mathcal{F}), \mathfrak{m}$ integral ideals of $F, \lambda \in \mathrm{Cl}^+(F)\}$ is a Λ-adic form of level $\mathfrak{n}$ and character χ if for all finitely many $k \geq 2$ there exists $f_k \in M_k(\mathfrak{n}_S, \chi\omega^{1-k}; E)$ such that $\nu_k(c(\mathfrak{m}, \mathcal{F})) = c(\mathfrak{m}, f_k), \nu_k(c_\lambda(, \mathcal{F})) = c_\lambda(0, f_k)$ is called a Λ-adic modular form.*

Furthermore, if $\nu_k(\mathcal{F})$ is in $S_k(\mathfrak{n}_S, \chi\omega^{1-k})$ for all but finitely many $k \geq 2$, then we say $\mathcal{F}$ is a Λ-adic cusp form.

The space of Λ-adic modular forms (resp. cusp forms) of level $\mathfrak{n}$ and character χ is denoted by $\mathcal{M}(\mathfrak{n}, \chi)$ (resp. $\mathcal{S}(\mathfrak{n}, \chi)$). By extension of scalars, the elements of $\mathcal{M}(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$ (resp. $\mathcal{S}(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$) are also called Λ-adic modular forms (resp. cusp forms).

The usual Hecke operators $T_\mathfrak{q}, U_\mathfrak{q}$ commute with specialisation. Thus, the action of these operators on the spaces $M_k(\mathfrak{n}_S, \chi\omega^{1-k}), S_k(\mathfrak{n}_S, \chi\omega^{1-k})$ give rise to action in the space $\mathcal{M}(\mathfrak{n}, \chi)$ that preserves $\mathcal{S}(\mathfrak{n}, \chi)$. We also define the ordinary subspaces

$$\mathcal{M}^o(\mathfrak{n}, \chi) := e\mathcal{M}(\mathfrak{n}, \chi), \quad \mathcal{S}^o(\mathfrak{n}, \chi) := e\mathcal{S}(\mathfrak{n}, \chi)$$

It is well known that the ordinary subspaces are finitely generated torsion-free Λ-modules. Let

$$\widetilde{\mathbf{T}} \subseteq \mathrm{End}(\mathcal{M}^o(\mathfrak{n}, \chi)), \quad \mathbf{T} \subseteq \mathrm{End}(\mathcal{S}^o(\mathfrak{n}, \chi))$$

be the Λ algebras generated by the Hecke operators $T_\mathfrak{q}, U_\mathfrak{q}$.
By extension of scalars, the elements of $\mathcal{M}^o(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$ (resp. $\mathcal{S}^o(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$) are also called Λ-adic modular forms (resp. cusp forms).

Proposition 55. *[DDP11, Proposition 3.2] If η, ψ is a pair of narrow ray class characters modulo $\mathfrak{m}_\eta, \mathfrak{m}_\psi$ respectively, such that $\eta\psi$ is totally odd. Then, there* *exists a* Λ*-adic modular form*

$$\mathcal{E} \in \mathcal{M}(\mathfrak{m}_\eta \mathfrak{m}_\psi, \eta\psi) \otimes_\Lambda \mathcal{F}_\Lambda$$

such that

$$\nu_k(\mathcal{E}(\eta, \psi)) = E_k(\eta, \psi\omega^{1-k})$$

Proof.

$$\begin{aligned} c(\mathfrak{m}, E_k(\eta, \psi\omega^{1-k})) &= \sum_{\mathfrak{r}|\mathfrak{m}} \eta(\mathfrak{m}/\mathfrak{r})\psi(\mathfrak{r})\omega^{1-k}(\mathfrak{r})\mathbb{N}\mathfrak{r}^{k-1} \\ &= \sum_{\substack{\mathfrak{r}|\mathfrak{m} \\ \gcd(\mathfrak{r},p)=1}} \eta(\mathfrak{m}/\mathfrak{r})\psi(\mathfrak{r})\langle \mathbb{N}\mathfrak{r}\rangle^{k-1} \end{aligned}$$

Moreover, if we choose $s \in \mathbb{Z}_p$ such that $\langle \mathbb{N}\mathfrak{r}\rangle = u^s$. Then,

$$\nu_k((1+T)^s) = \langle \mathbb{N}\mathfrak{r}\rangle^{k-1}$$

Thus, the terms on the right hand side can be seen as specialisation of elements in Λ. Moreover, the $L_{S,p}(\eta^{-1}\psi\omega, 1-k)$ can also be seen as specialisation of an element of Λ. Hence, $E_k(\eta, \psi\omega^{1-k})$ can be seen as a specialisation of an appropriate Λ-adic form. This completes the proof. ☐

Definition 56 (Shifted weight forms). *A family* $\mathcal{M}' = \{c(\mathfrak{m}, \mathcal{F}), c_\lambda(0, \mathcal{F}), \mathfrak{m}$ *integral ideals of* $F, \lambda \in \mathrm{Cl}^+(F)\}$ *is a* Λ*-adic form of level* $\mathfrak{n}$ *and character* χ *if for all finitely many* $k \geq 2$ *there exists* $f_k \in M_{k-1}(\mathfrak{n}_S, \chi\omega^{1-k}; E)$ *such that* $\nu_k(c(\mathfrak{m}, \mathcal{F})) = c(\mathfrak{m}, f_k), \nu_k(c_\lambda(, \mathcal{F})) = c_\lambda(0, f_k)$ *is called a* Λ*-adic modular form.*

Proposition 57. *There exists an element* $\mathcal{G} \in \mathcal{M}' \otimes_\Lambda \mathcal{F}_\Lambda$ *such that*

$$\nu_k(\mathcal{G}) = G_{k-1}(1, \omega^{1-k})$$

Furthermore, if Leopoldt's conjecture holds for F*, then the form* $\mathcal{G} \in \mathcal{M}' \otimes_\Lambda \Lambda_{(1)}$*, and*

$$\nu_1(\mathcal{G}) = 1$$

Proof. The existence of $\mathcal{G}$ follows by defining it via

$$\nu_k(c(\mathfrak{m}, \mathcal{G})) = 2^n \zeta_p(F, 2-k)^{-1} \sum_{\substack{\mathfrak{r}|\mathfrak{m} \\ \gcd(\mathfrak{r},p)=1}} \eta(\mathfrak{m}/\mathfrak{r})\psi(\mathfrak{r})\langle \mathbb{N}\mathfrak{r}\rangle^{k-1}, \quad \nu_k(c_\lambda(0, \mathcal{G})) = 1$$

If Leopoldt's conjecture holds, then by a result of Colmez cite Colmez, the p-adic zeta-function $\zeta_p(F, s)$ has a pole at $s = 1$ and thus $\zeta_p(F, 2-k)^{-1}$ is regular at $s = 1$ and vanishes at that point. This completes the proof. ☐

It is crucial to observe that this is the place where we use the Leopoldt conjecture assumption in theorem 34 to show the existence of such a $\mathcal{G}$. The Leopoldt conjecture was removed by Ventullo in his thesis [Ven14]. He showed that:

THEOREM 58. *There exists a Hida family* $\mathcal{G} \in \mathcal{M}^o(1, \omega^{-1}) \otimes_\Lambda \Lambda_{(0)}$ *with the property that* $\nu_0(\mathcal{G}) = 1$ *and* $c_\lambda(0, \mathcal{G}) = 1$ *for all* $\lambda \in \mathrm{Cl}^+(F)$.

We set

$$G_k := \nu_k(\mathcal{G}) \in M_k(p, \omega^{-k})$$

Notice that we no more have any control over what the $\mathcal{G}$ looks like like we earlier did. This is going to present a serious challenge in what is to follow.

Remark 59. *If you came from the previous chapter on constructing the cusp-form, now is the time to go back.*

2. Λ-adic cusp form

Very naturally, we want to know if our classical modular forms P_k, P_k^o, H_k and the cusp form F_k can be interpolated p-adically. This is where the slightly ad-hoc condition in the hypothesis comes in handy.

Proposition 60. *[DDP11, Proposition 3.4, Lemma 3.5] Suppose Leopoldt's conjecture holds for* F*, and*

$$\mathrm{ord}_{k=1}(\mathcal{L}_{an}(\chi, k) + \mathcal{L}_{an}(\chi^{-1}, k)) = \mathrm{ord}_{k=1} \mathcal{L}_{an}(\chi^{-1}, k)$$

Then there exist Λ*-adic forms* $\mathcal{P} \in \mathcal{M}(\mathfrak{n}, \chi) \otimes \Lambda_{(1)}, \mathcal{P}^o, \mathcal{H} \in \mathcal{M}^o(\mathfrak{n}, \chi) \otimes \Lambda_{(1)}, \mathcal{F} \in \mathcal{S}^o(\mathfrak{n}, \chi) \otimes \Lambda_{(1)}$ *such that for all* $k \geq 2$

$$\nu_k(\mathcal{P}) = P_k, \nu_k(\mathcal{P}^o) = P^o, \nu_k(\mathcal{H}) = H_k, \nu_k(\mathcal{F}) = F_k$$

In particular, the weight 1 *specialisations are*

$$\nu_1(\mathcal{P}) = \nu_1(\mathcal{P}^o) = E_1(1, \chi_R), \nu_1(\mathcal{H}) = E_1(1, \chi_S),$$

$$\nu_1(\mathcal{F}) = tE_1(1, \chi_S) \text{ for some } t \in E^\times$$

PROOF. Set

$$\mathcal{P} = E_1(1, \chi_R)\mathcal{G}, \mathcal{P}^o = e\mathcal{P}$$

Here, the $\mathcal{G}$ is the one created explicitly under the assumption of Leopoldt conjecture (NOT the one due to Ventullo).

To define H_k recall that we had the coefficients u_k, v_k, w_k. They can themselves be viewed as specialisations of $u, v, w \in \mathcal{F}_\Lambda$ such that $u(k) = u_k, v(k) = v_k, w(k) = w_k$ for all but finitely many $k \geq 2$. Using the technical condition stated in the hypothesis, we can show that $u, v, w \in \Lambda_{(1)}$ and u is invertible in this ring. Then, we set

$$\mathcal{H} := u\mathcal{E}(1, \chi) + v\mathcal{E}(\chi, 1) + v\mathcal{P}^o$$

which belongs to $\mathcal{M}^0(\eta, \chi) \otimes \Lambda_{(1)}$ and the specialisation $\nu_k(\mathcal{H}_k) = H_k$.

Note that the Hecke operators defined in the previous chapter can be viewed as elements of the ordinary Λ-adic Hecke algebra $\widetilde{\mathbf{T}}$. If we set

$$\mathcal{F} := \prod_{(\eta,\psi)} T_{(\eta,\psi)}\mathcal{H}$$

we obtain the desired Λ-adic cusp form with specialisation $\nu_k(\mathcal{F}) = F_k$. □

Next, we wish to interpolate the semi-cusp forms constructed in previous chapter.

Recall the two semi-cusp forms:

(36)

$$H_k := E_k(1, \chi\omega^{1-k}) - E_1(1, \chi_{R'})G_{k-1}\frac{L_p(\chi\omega, 1-k)}{L(\chi_{R'}, 0)}$$

(37)

$$H_k := E_k(1, \chi\omega^{1-k}) - E_1(1, \chi)G_{k-1}\frac{L_p(\chi\omega, 1-k)}{L(\chi, 0)} + E_k(\chi, \omega^{1-k})\frac{L_p(\chi\omega, 1-k)}{L(\chi, 0)}\frac{L(\chi^{-1}, 0)}{L_p(\chi^{-1}\omega, 1-k)}$$

where the first one is when $R' \neq \emptyset$ and second one is when $R' = \emptyset$.

We want to interpolate these forms to appropriate Hida families. First, note that $T \mapsto (1+T)u^{-1} - 1$ has the effect of shifting the specialisation map from k to $k-1$. More precisely, we have

$$\nu_k(\mathcal{G}((1+T)u^{-1} - 1)) = \nu_k(\mathcal{G}(T))$$

In case 1, i.e. when $R' \neq \emptyset$, we have the Λ-adic family:

$$\mathcal{H} = \mathcal{E}(1, \chi) - E_1(1, \chi_{R'})\mathcal{G}((1+T)u^{-1} - 1)\frac{\mathcal{L}(\chi\omega)}{L(\chi_{R'}, 0)} \tag{38}$$

where $\mathcal{L}(\chi\omega) \in \Lambda_{(1)}$ is the (Λ-adic) p-adic L-function whose specialisation is $\nu_k(\mathcal{L}(\chi\omega)) = L_p(\chi\omega, 1-k)$. Hence, $\nu_k(\mathcal{H}) = H_k$ for all positive integers k in the neighbourhood of $1 \in \mathbb{Z}_p$.

When $R' = \emptyset$, we set

$$\mathcal{H} = \mathcal{E}(1, \chi) - E_1(1, \chi)\mathcal{G}((1+T)u^{-1} - 1)\frac{\mathcal{L}(\chi\omega)}{L(\chi, 0)} + \mathcal{E}(\chi, 1)\mathcal{W} \tag{39}$$

where

$$\mathcal{W} := \frac{\mathcal{L}(\chi\omega)}{\mathcal{L}(\chi^{-1}\omega)}\frac{L(\chi^{-1})}{L(\chi, 0)} \in \mathrm{Frac}(\Lambda)$$

has specialisation

$$\nu_k(\mathcal{W}) = \frac{L_p(\chi\omega, 1-k)}{L_p(\chi^{-1}\omega, 1-k)}\frac{L(\chi^{-1}, 0)}{L(\chi, 0)}$$

for every $k \in \mathbb{Z}_p$ with $L_p(\chi^{-1}\omega, 1-k) \neq 0$.

We require that $\mathcal{W}$ has no pole at weight 1, which happens if

$$\operatorname{ord}_\pi \mathcal{W} = r_{an}(\chi) - r_{an}(\chi^{-1}) < 0$$

where π is an uniformiser of the discrete valuation ring $\Lambda_{(1)}$. We choose π to be $T/\log u$ for computational convenience. And,

$$r_{an} = r_{an}(\chi) := \operatorname{ord}_{s=0} L_p(\chi\omega, s)$$

If we swap χ for χ^{-1}, then we end up inverting $\mathcal{W}$. Hence, when $\mathcal{W}$ has a pole at $k = 1$, we can assume $\mathcal{W}$ has a zero at $k = 1$ and instead show the conjecture for χ^{-1}. We can therefore assume that $\operatorname{ord}_\pi \mathcal{W} \geq 0$ and subdivide the case $R' = \emptyset$ into further two cases:

Case-2: $\nu_1(\mathcal{W}) \neq 0$: Here, we must prove $\mathcal{R}_p(\chi) = \mathcal{L}_{an}(\chi)$,

Case-3: $\nu_1(\mathcal{W}) = 0$: Here, we must show that $\mathcal{R}_p(\chi) = \mathcal{L}_{an}(\chi) = 0$ and $\mathcal{R}_p(\chi^{-1}) = \mathcal{L}_{an}(\chi^{-1})$

Now that we have our semi-cusp forms, we will just record a theorem proved in DDP that demonstrates how to get a cusp form.

THEOREM 61. *There exists a Hecke operator* t *in the Hecke algebra* $\widetilde{\mathbf{T}}_{(1)}$ *such that* $\mathcal{F} = t \cdot e \cdot \mathcal{H}$ *is a cuspidal* Λ*-adic form lying in* $\mathcal{S}^0(\eta, \chi)_{(1)}$.

CHAPTER 5

Hida Algebra Homomorphism

1. $1+\epsilon$ specialisation

Let $\nu_{1+\epsilon} : \Lambda_{(1)} \mapsto \widetilde{E} := E[\epsilon]/(\epsilon^2)$ be the map $f \mapsto f(1) + f'(1)\epsilon$.

Recall that $\phi(1)(T_\mathfrak{q}) = \nu_1(c(\mathfrak{q}, T_\mathfrak{q}\mathcal{H})) = T_\mathfrak{q} H_1 = T_\mathfrak{q} E_1(1, \chi_S) = 1 + \chi_S(\mathfrak{q})$ for $\mathfrak{q} \neq \mathfrak{p}$. The observation is that $H_{1+\epsilon} = \nu_{1+\epsilon}(\mathcal{H})$ can also be written as sum of two characters that lift $1, \chi$.

Definition 62. *Let $\psi_1 : G_F \to \widetilde{E}$ be a character unramified outside p and defined by*

$$\psi_1(\mathfrak{q}) = 1 + \nu_1 \kappa_{cyc}(\mathfrak{q})\epsilon \quad \forall\ \mathfrak{q} \nmid p$$

$$\psi_1(\mathfrak{q}) = 1 \quad \mathfrak{q} \mid p$$

Let $\psi_2 : G_F \to \widetilde{E}$ be a character unramified outside S and defined by

$$\psi_2(\mathfrak{q}) = \chi(\mathfrak{q})(1 + u_1 \kappa_{cyc}(\mathfrak{q})\epsilon \quad \forall\ \mathfrak{q} \nmid p$$

$$\psi_2(\mathfrak{q}) = 0 \quad \mathfrak{q} \in S$$

Theorem 63. *[DDP11, Proposition 3.6] The Fourier coefficients of $H_{1+\epsilon}$ satisfy*

(1) $c(1, H_{1+\epsilon}) = 1$
(2) $c(\mathfrak{q}, H_{1+\epsilon}) = \psi_1(\mathfrak{q}) + \psi_2(\mathfrak{q})$ *if* $\mathfrak{q} \neq \mathfrak{p}$
(3) $c(\mathfrak{p}, H_{1+\epsilon}) = 1 + w_1'(\epsilon)$

And, $H_{1+\epsilon}$ is a simultaneuous eigenform for the Hecke operators $T_\mathfrak{q}$ for $\mathfrak{q} \notin S$ and $U_\mathfrak{q}$ for $\mathfrak{q} \in S$. The eigenvalues are given by the above calculated coefficients.

This lets us define a $\Lambda_{(1)}$ homomorphism

$$\phi_{1+\epsilon} : \widetilde{\mathbf{T}} \otimes \Lambda_{(1)} \to \widetilde{E}, \quad T \mapsto \nu_{1+\epsilon}(c(\mathcal{O}_F, T\mathcal{H}))$$

In fact, $\phi_{1+\epsilon}$ factors through the quotient $\mathbf{T} \otimes \Lambda_{(1)}$ of $\widetilde{\mathbf{T}} \otimes \Lambda_{(1)}$ as there is a Hecke operator T such that $T\mathcal{H} = \mathcal{F}$.

Proof. We shall prove this theorem in the case when $\mathfrak{p}$ is not the only prime in H above p (the other case is done in [DDP11, Proposition 3.6]). Let $\mathfrak{m}$ be an integral ideal of $\mathcal{O}_F$ and write $\mathfrak{m} = \mathfrak{n}\langle p \rangle$ with $\gcd(\mathfrak{n}, \langle p \rangle) = 1$. Note that

$$c(\mathfrak{m}, E_{1+\epsilon}(1, \chi)) = \sum_{\mathfrak{r} \mid \mathfrak{n}} \chi(\mathfrak{r})(1 + \epsilon \kappa_{cyc}(\mathfrak{r})) \tag{40}$$

$$\chi(\mathfrak{r}) = \chi_S(\mathfrak{r}) \qquad \text{if } \mathfrak{p} \nmid \mathfrak{r} \tag{41}$$

$$\chi(\mathfrak{r}) = 0 \qquad \text{if } \mathfrak{p} \mid \mathfrak{r} \tag{42}$$

Therefore,

$$c(\mathfrak{m}, E_1(1,\chi)) - c(\mathfrak{m}, E_1(1,\chi_S)) = \sum_{\mathfrak{r}|\mathfrak{m}} (\chi(\mathfrak{r}) - \chi_S(\mathfrak{r})) = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) \sum_{\mathfrak{r}|\mathfrak{n}} \chi(\mathfrak{r}) \tag{43}$$

Using the same arguments as in [DDP11] we can show that

$$c(\mathfrak{m}, H_{1+\epsilon}) = \left(\sum_{\mathfrak{r}|\mathfrak{n}} \psi_1(\mathfrak{n}/\mathfrak{r})\psi_2(\mathfrak{r}) \right) (1 + w_1'\epsilon)^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}$$

The result follows from this. □

As $w(k) = u(k)\mathcal{L}_{an}(\chi, k)$ we have $w_1' = u_1\mathcal{L}_{an}(\chi)$. Thus, we have the following

THEOREM 64. *[DDP11, Theorem 3.7] Assuming that Leopoldt's conjecture holds for F, and the assumptions*

(1) *If $|S_p| > 1$, then the conjecture is true for all χ.*
(2) *If $|S_p| = 1$ and furthermore*

$$\mathrm{ord}_{k=1}(\mathcal{L}_{an}(\chi, k) + \mathcal{L}_{an}(\chi^{-1}, k)) = \mathrm{ord}_{k=1} \mathcal{L}_{an}(\chi^{-1}, k) \tag{44}$$

Then there exists a $\Lambda_{(1)}$-homomorphism

$$\phi_{1+\epsilon} : \mathbf{T}_{(1)} \to \widetilde{E}$$

such that

$$\phi_{1+\epsilon}(T_{\mathfrak{q}}) = \psi_1(\mathfrak{q}) + \psi_2(\mathfrak{q}) \quad \mathfrak{q} \notin S \tag{45}$$

$$\phi_{1+\epsilon}(U_{\mathfrak{q}}) = \psi_1(\mathfrak{q}) \quad \mathfrak{q} \in R \tag{46}$$

$$\phi_{1+\epsilon}(U_{\mathfrak{p}}) = 1 + u_1\mathcal{L}_{an}(\chi)\epsilon \tag{47}$$

2. General rank, case 1: $R' \neq \emptyset$

As seen in the previous section, in a small neighbourhood of 1, the form $\mathcal{F}$ remains an eigenform. This allows us to define a Λ-algebra homomorphism

$$\phi : \mathbf{T} \to E[T]/\langle T^2 \rangle$$

sending t to $\nu_1(t \cdot \mathcal{F})$.

In the general case, one attempt could have been to construct a Λ-algebra homomorphism

$$\mathbf{T} \to E[T]/T^{r+1}$$

It so turns out that the cuspidal form $\mathcal{F}$ no longer remains an eigenform modulo T^{r+1}. Ventullo's crucial insight was to simply study the orbit of this form $\mathcal{F}$ modulo T^{r+1} under the Hecke action. We will see that the orbit is not one dimensional over Λ/T^{r+1} but it is finite dimensional and can be explicitly obtained. This is the objective of this chapter and the following few sections.

Any Hida family is determined exclusively by its Fourier expansion, there is a canonical homomorphism:

$$c : \mathcal{S}^0(\eta, \chi)_{(1)} \to \prod_{\mathfrak{a} \subseteq \mathcal{O}_F} \Lambda_{(1)} \tag{48}$$

$$\mathcal{X} \mapsto (c(\mathfrak{a}, \mathcal{X}))_{\mathfrak{a} \subseteq \mathcal{O}_F} \tag{49}$$

Let $\mathcal{H}$ be the image of $\mathcal{F}$ under the reduction of c modulo $\pi^{r_{an}+1}$. This is finitely generated module over $\Lambda_{(1)}/\pi^{r_{an}+1} = E[\pi]/\pi^{r_{an}+1}$, and hence we have a canonical Λ-algebra homomorphism:

$$\phi : \mathbf{T} \to \mathrm{End}_{E[\pi]/\pi^{r_{an}+1}} \mathcal{H}$$

Now, we come to the construction of the alluded Hecke algebra homomorphism.

THEOREM 65. *Suppose R' is not empty. Then, there exists a Λ-algebra homomorphism*

$$\phi : \mathbf{T} \to W_1 = E[\pi, \epsilon_1, \dots, \epsilon_r]/\langle \pi^{r_{an}+1}, \epsilon_i^2, \epsilon_i \pi, \epsilon_1 \cdots \epsilon_r + (-1)^{r_{an}} \mathcal{L}^*_{an}(\chi) \pi^{r_{an}} \rangle$$

where

$$\mathcal{L}^*_{an} := \frac{L_p^{(r_{an})}(\chi, 0)}{r_{an}! L(\chi_{R'}, 0)}$$

such that

$$\begin{aligned} T_\mathfrak{l} &\mapsto 1 + \chi\varepsilon(\mathfrak{l}) & \mathfrak{l} &\nmid \mathfrak{n}p, \\ U_\mathfrak{l} &\mapsto 1 & \mathfrak{l} &\mid \mathfrak{n} \text{ or } \mathfrak{l} \in R', \\ U_{\mathfrak{p}_i} &\mapsto 1 + \epsilon_i & R &= \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\} \end{aligned}$$

Here, ε is the Λ-adic cyclotomic character that we have not yet introduced. We will do that here. The Λ-adic cyclotomic character

$$\varepsilon : G_F \to \Lambda^\times$$

satisfies $\nu_k(\varepsilon(\sigma)) = \langle \varepsilon_{cyc}(\sigma) \rangle^{k-1}$ for any $k \in \mathbb{Z}_p$ where ε_{cyc} is the usual cyclotomic character. Thus, explicitly, the Λ-adic cyclotomic character is given by

$$\varepsilon(\sigma) = (1+T)^{\log_p \langle \varepsilon_{cyc}(\sigma) / \log_p u \rangle}$$

2.1. Proof of theorem 65. The second term in $\mathcal{F}$ is

$$\mathcal{F}' = E_1(1, \chi_{R'}) \mathcal{G}((1+T)u^{-1} - 1) \frac{\mathcal{L}(\chi\omega)}{L(\chi_{R'}, 0)}$$

Suppose

$$\mathcal{L}(\chi\omega) = \pi^{r_{an}}(a_0 + a_1 T + \cdots + a_n T^n + \cdots) \tag{50}$$

As $\mathcal{L}(\chi\omega, u^{k-1} - 1) = L_p(\chi\omega, 1-k)$ for positive integers in the neighbourhood of $1 \in \mathbb{Z}_p$, we have $\mathcal{L}(\chi\omega, u^{-s}-1) = L_p(\chi\omega, s)$ for $s \in \mathbb{Z}_p$. Now, using this information, we can compute eq. (50) at $u^{-s} - 1$ to get

$$L_p(\chi\omega, s) = \mathcal{L}(\chi\omega, u^{-s} - 1) = \left(\frac{u^{-s}-1}{\log u}\right)^{r_{an}} \left(a_0 + a_1\left(\frac{u^{-s}-1}{\log u}\right) + \cdots + a_n\left(\frac{u^{-s}-1}{\log u}\right)^n + \cdots\right) \tag{51}$$

Differentiating gives

$$\begin{aligned} L_p^{(r_{an})}(\chi\omega, s) = (-1)^{r_{an}} r_{an}! \left(a_0 + a_1\left(\frac{u^{-s}-1}{\log u}\right) + \cdots + a_n\left(\frac{u^{-s}-1}{\log u}\right)^n + \cdots\right) \\ +(\text{ terms containing the factor } (u^{-s}-1)) \end{aligned} \tag{52}$$

And, evaluating at $s = 0$ gives

$$L_p^{(r_{an})}(\chi\omega, s) = (-1)^{r_{an}} r_{an}! a_0 \tag{53}$$

Using eq. (53), the form $\mathcal{F}'$ modulo $\pi^{r_{an}+1}$ becomes

$$\begin{aligned} \mathcal{F}' &\equiv E_1(1, \chi_{R'}) \cdot 1 \cdot \pi^{r_{an}} \cdot a_0 \cdot \frac{1}{L(\chi_{R'}, 0)} \pmod{\pi^{r_{an}+1}} \\ &\equiv E_1(1, \chi_{R'}) \cdot \pi^{r_{an}} \cdot \frac{(-1)^{r_{an}} L_p^{(r_{an})}(\chi\omega, 0)}{r_{an}!} \cdot \frac{1}{L(\chi_{R'}, 0)} \pmod{\pi^{r_{an}+1}} \\ &\equiv (-1)^{r_{an}} E_1(1, \chi_{R'}) \mathcal{L}_{an}^* \pi^{r_{an}} \pmod{\pi^{r_{an}+1}} \end{aligned} \tag{54}$$

Thus, the Hecke orbit of $\mathcal{F}'$ depends on the Hecke orbit of $E_1(1, \chi_{R'})$. In particular, if $t \in \widetilde{\mathbf{T}}$, then

$$t\mathcal{F}' \equiv (-1)^{r_{an}} \nu_1(t)(E_1(1, \chi_{R'}))\mathcal{L}_{an}^* \pi^{r_{an}} \pmod{\pi^{r_{an}+1}}$$

Thus, we should analyse the action of the Hecke operators on the Eisenstein series $E_1(1, \chi_{R'})$. We recall that

$$T_\mathfrak{l} E_1(1, \chi_{R'}) = (1 + \chi(\mathfrak{l}))E_1(1, \chi_{R'}) \qquad \mathfrak{l} \nmid \mathfrak{n}p, \tag{55}$$

$$U_\mathfrak{l} E_1(1, \chi_{R'}) = E_1(1, \chi_{R'}) \qquad \mathfrak{l} \mid \mathfrak{n} \text{ or } \mathfrak{l} \in R' \tag{56}$$

whereas for $\mathfrak{p} \in R$ we have

$$U_\mathfrak{p} E_1(1, \chi_{R'}) = E_1(1, \chi_{R'}) + E_1(1, \chi_{R' \cup \{\mathfrak{p}\}})$$

In general, if $R' \subseteq J \subseteq S_p$ and $\mathfrak{p} \in S_p$ we have

$$(U_{\mathfrak{p}} - 1)E_1(1, \chi_J) = \begin{cases} E_1(1, \chi_{J \cup \{\mathfrak{p}\}}) & \mathfrak{p} \notin J, \\ 0 & \mathfrak{p} \in J \end{cases} \tag{57}$$

When $\mathfrak{l} \nmid \mathfrak{n}p$, we know that $T_{\mathfrak{l}}\mathcal{E}(1,\chi) = (1 + \chi\varepsilon(\mathfrak{l}))\mathcal{E}(1,\chi)$. From the definition of the Λ-adic cyclotomic character we also have

$$1 + \chi\varepsilon(\mathfrak{l}) \equiv 1 + \chi(\mathfrak{l}) \pmod{\pi}$$

Hence,

$$T_{\mathfrak{l}}\mathcal{H} = (1 + \chi\varepsilon(\mathfrak{l}))\mathcal{H} \qquad \mathfrak{l} \nmid \mathfrak{n}p, \tag{58}$$

$$U_{\mathfrak{l}}\mathcal{H} = \mathcal{H} \qquad \mathfrak{l} \mid \mathfrak{n} \text{ or } \mathfrak{l} \in R' \tag{59}$$

By the commutativity of Hecke operators, the same is true for $\mathcal{F}$ and its Hecke orbit $\mathcal{H}$. Hence, the homomorphism ϕ takes

$$\phi(T_{\mathfrak{l}}) = (1 + \chi\varepsilon(\mathfrak{l})) \qquad \mathfrak{l} \nmid \mathfrak{n}p, \tag{60}$$

$$\phi(U_{\mathfrak{l}}) = 1 \qquad \mathfrak{l} \mid \mathfrak{n} \text{ or } \mathfrak{l} \in R' \tag{61}$$

Next, $U_{\mathfrak{p}_i}\mathcal{E}(1,\chi) = \mathcal{E}(1,\chi)$ OR $U_{\mathfrak{p}_i} - 1$ annihilates $\mathcal{E}(1,\chi)$. From eq. (54), we see that π annihilates the image of $(U_{\mathfrak{p}_i} - 1)\mathcal{F}$ in $\mathcal{H}$. And, from eq. (54), eq. (57) we see that the image of $(U_{\mathfrak{p}_i} - 1)^2\mathcal{F}$ is 0 in $\mathcal{H}$. If we let $\phi(U_{\mathfrak{p}_i}) = 1 + \epsilon_i$, then from our observations, it is clear that

$$\epsilon_i^2 = 0, \pi \cdot \epsilon_i = 0 \text{ for all } i \tag{62}$$

This almost finishes the search for the relations. We just have the last one. For that, observe that

$$\prod_{i=1}^{r}(U_{\mathfrak{p}_i} - 1)\mathcal{F} \equiv t \cdot e((-1)^{r_{an}} E_1(1, \chi_S)\mathcal{L}^*_{an}\pi^{r_{an}}) \pmod{\pi^{r_{an}+1}} \tag{63}$$

$$\equiv t \cdot e((-1)^{r_{an}}\mathcal{L}^*_{an}\pi^{r_{an}}\mathcal{E}(1,\chi)) \pmod{\pi^{r_{an}+1}} \tag{64}$$

$$= (-1)^{r_{an}}\mathcal{L}^*_{an}\pi^{r_{an}}\mathcal{F} \pmod{\pi^{r_{an}+1}} \tag{65}$$

Thus, we have the relation

$$\epsilon_1 \cdots \epsilon_r + (-1)^{r_{an}}\mathcal{L}^*_{an}(\chi)\pi^{r_{an}}$$

in $\mathrm{End}_{E[\pi]/\pi^{r_{an}+1}}\mathcal{H}$. Combining these relations, we get a surjective $\Lambda_{(1)}$ algebra homomorphism

$$W_1 \to \phi(\mathbf{T}) \otimes E$$

To finish the proof we need to show that this map is injective as well. That is done by counting dimensions both sides. Notice that W_1 has dimension $2^r + r_{an} - 1$ as E-vector space, generated by $1, \pi, \pi^2, \ldots, \pi^{r_{an}-1}$ and the products $\prod_{j\in J} \epsilon_j$ for all non-empty subsets $J \subseteq R$. We can show that that the corresponding $1, \pi^1, \pi^2, \ldots, \pi^{r_{an}-1}$ and the products $\prod_{j\in J} \epsilon_j$ are linearly independent in the endomormphism ring $\mathrm{End}_{E[\pi]/\pi^{r_{an}+1}} \mathcal{H}$. To show this, it is enough to show that the images of $\mathcal{F}$ under the operators are linearly independent. As $\mathcal{F}, \pi\mathcal{F}, \pi^2\mathcal{F}, \ldots, \pi^{r_{an}-1}\mathcal{F}$ vanish to different orders less than r_{an}, the coefficients of these terms in any linear combination have to be zero. And, modulo $\pi^{r_{an}+1}$ the forms $\prod_{j\in J} \epsilon_j\mathcal{F}$ are some multiple of $E_1(1, \chi_{R'\cup J})\pi^{r_{an}}$ which we know are linearly independent. This finishes the proof.

3. General rank, case $2 : R' = \emptyset, \nu_1(\mathcal{W}) = 0$

We assume that $\mathcal{W}$ has a zero at $k = 1$, i.e. $r_{an}(\chi) > r_{an}(\chi^{-1})$. Let $s := r_{an}(\chi), t := r_{an}(\chi^{-1})$. Then, we can say that

$$\mathrm{ord}_\pi \mathcal{W} = s - t \geq 1$$

We define the $\Lambda_{(1)}$-algebra

$$E[\pi, \epsilon_1, \ldots, \epsilon_r, y]/I_{W_2}$$

where

$$I_{W_2} = \langle \pi^s, y^{t+1}, y(\pi - y), \pi^t\mathcal{W} - y^t, \epsilon_i^2, \epsilon_i\pi, \epsilon_i y, \epsilon_1 \cdots \epsilon_r + (-1)^s\mathcal{L}_{an}^*(\chi)\pi^s\rangle$$

Theorem 66. *We have a Λ-algebra homomorphism*

$$\phi : \mathbf{T} \to W_2$$

such that

$$\begin{aligned}
T_l &\mapsto 1 + \chi\varepsilon(l) + \frac{(\chi(l) - 1)(1 - \varepsilon(l))}{\pi} y & & l \nmid np,\\
U_l &\mapsto 1 + \frac{1 - \varepsilon(l)}{\pi} y & & l \mid n \textit{ or } l \in R',\\
U_{p_i} &\mapsto 1 + \epsilon_i & & R = \{p_1, \ldots, p_r\}
\end{aligned}$$

3.1. Proof of theorem 66. Like in the previous proof, we will first construct a homomorphism

$$W_2 \to \phi(\mathbf{T}) \otimes E$$

and show that the above homomorphism is in fact an isomorphism.

First, fix a prime $q \nmid np$ such that $\chi(q) \neq 1$. Define the Hecke operator

$$Y := \frac{T_q - 1 - \chi\varepsilon(q)}{(1 - \varepsilon(q))(\chi(q) - 1)/\pi} \in \widetilde{\mathbf{T}}$$

It can be shown that

$$\begin{aligned}\pi(T_q - 1 - \chi\varepsilon(q))\mathcal{H} = \Big[&\pi(T_q - 1 - \chi\varepsilon(q))\mathcal{E}(1,\chi) - (T_q - 1 - \chi\varepsilon(q))E_1(1,\chi)\mathcal{G}((1+T)u^{-1} - 1)\frac{\mathcal{L}(\chi\omega)}{L(\chi,0)} \\ &+ \pi(T_q - 1 - \chi\varepsilon(q))\mathcal{E}(\chi,1)\Big] \\ &\equiv \pi\,[(1+\chi\varepsilon(q))\mathcal{E}(1,\chi) - (1+\chi\varepsilon(q))\mathcal{E}(1,\chi)] - 0 + \pi\,[(\chi(q)+\varepsilon(q))\mathcal{E}(\chi,1) \\ &- (1+\chi\varepsilon(q))\mathcal{E}(\chi,1)\,]\,\mathcal{W} \pmod{\pi^{r_{an}+1}} \\ &\equiv \pi(\chi(q)-1)(1-\varepsilon(q))\mathcal{E}(\chi,1)\mathcal{W} \pmod{\pi^{r_{an}+1}} \qquad (66)\end{aligned}$$

Hence,

$$Y\mathcal{H} \equiv \pi\mathcal{E}(\chi,1)\mathcal{W} \pmod{\pi^{r_{an}+1}} \tag{67}$$

Therefore,

$$\begin{aligned}T_l\mathcal{H} &= (1+\chi\varepsilon(l))\mathcal{E}(1,\chi) - (1+\chi(l))E_1(1,\chi)\mathcal{G}\frac{\mathcal{L}(\chi\omega)}{L(\chi,0)} + (\chi(l)+\varepsilon(l))\mathcal{E}(\chi,1)\mathcal{W} \\ &\equiv (1+\chi\varepsilon(l))\mathcal{E}(1,\chi) - (1+\chi\varepsilon(l))E_1(1,\chi)\mathcal{G}\frac{\mathcal{L}(\chi\omega)}{L(\chi,0)} + (\chi(l)+\varepsilon(l))\mathcal{E}(\chi,1)\mathcal{W} \\ &+ (1+\chi\varepsilon(l))\mathcal{E}(\chi,1)\mathcal{W} - (1+\chi\varepsilon(l))\mathcal{E}(\chi,1) \pmod{\pi^{r_{an}+1}} \\ &\equiv (1+\chi\varepsilon(l))\mathcal{H} + (\chi(l)-1)(1-\varepsilon(l))\mathcal{E}(\chi,1)\mathcal{W} \pmod{\pi^{r_{an}+1}} \\ &\equiv (1+\chi\varepsilon(l))\mathcal{H} + \frac{(\chi(l)-1)(1-\varepsilon(l))}{\pi}Y\mathcal{H} \pmod{\pi^{r_{an}+1}} \qquad (68)\end{aligned}$$

As a consequence,

$$\phi(T_l) = 1 + \chi\varepsilon(l) + \frac{(\chi(l)-1)(1-\varepsilon(l))}{\pi}Y$$

for $l \nmid np$.

Similarly, as $\chi(l) = 0$ for $l \mid n$, we have

$$U_l\mathcal{F} = \left(1 + \frac{\varepsilon(l)-1}{\pi}Y\right)\mathcal{F} \pmod{\pi^{r_{an}+1}}$$

and

$$\phi(U_l) = 1 + \frac{\varepsilon(l)-1}{\pi}Y$$

Again, as $T_q\mathcal{E}(\chi,1) = (\chi(q)+\varepsilon(q))\mathcal{E}(\chi,1)$ we have

$$Y\mathcal{E}(\chi,1) = \pi\mathcal{E}(\chi,1) \tag{69}$$

Also,

$$\pi Y\mathcal{E}(\chi,1) \equiv \pi^2\mathcal{E}(\chi,1) \equiv Y^2\mathcal{E}(\chi,1) \pmod{\pi^{r_{an}+1}}$$
$$Y^{r_{an}+1}\mathcal{E}(\chi,1) \equiv 0 \pmod{\pi^{r_{an}+1}}$$

Consequently, we get

$$(\pi Y - Y^2)\mathcal{F} \equiv 0 \pmod{\pi^{r_{an}+1}} \tag{70}$$

$$Y^{r_{an}+1}\mathcal{F} \equiv 0 \pmod{\pi^{r_{an}+1}} \tag{71}$$

Now, using eq. (70)

$$\pi^{r_{an}}\mathcal{W}\mathcal{F} \equiv \pi^{r_{an}}\mathcal{W}\mathcal{E}(1,\chi) + \pi^{r_{an}}\mathcal{E}(\chi,1)\mathcal{W}^2 \pmod{\pi^{r_{an}+1}} \tag{72}$$

$$Y^{r_{an}}(\mathcal{W}+1)\mathcal{F} \equiv (\mathcal{W}+1)\pi^{r_{an}}\mathcal{E}(\chi,1)\mathcal{W} \pmod{\pi^{r_{an}+1}} \tag{73}$$

$$(\pi^{r_{an}}\mathcal{W} - Y^{r_{an}}(\mathcal{W}+1))\mathcal{F} \equiv 0 \pmod{\pi^{r_{an}+1}} \tag{74}$$

Where the last equation holds because $\mathcal{E}(1,\chi) \equiv \mathcal{E}(\chi,1) \equiv E_1(1,\chi_S) \pmod{\pi}$.

Next, we move on to the tricky operators $U_{\mathfrak{p}}$ where $\mathfrak{p} \in R$. Like in the proof in Case 1, we have

$$(U_{\mathfrak{p}} - 1)^2\mathcal{F} \equiv \pi(U_{\mathfrak{p}} - 1)\mathcal{F} \equiv 0 \pmod{\pi^{r_{an}+1}} \tag{75}$$

$$(U_{\mathfrak{p}} - 1)Y\mathcal{F} \equiv 0 \pmod{\pi^{r_{an}+1}} \tag{76}$$

For the last relation in I_{W_2}, we again compute

$$\begin{aligned}\prod_{i=1}^{r}(U_{\mathfrak{p}_i} - 1)\mathcal{H} &\equiv (-1)^{r_{an}+1}E_1(1,\chi_S)\mathcal{L}^*_{an}(\chi)\pi^{r_{an}} \pmod{\pi^{r_{an}+1}}\\ &\equiv (-1)^{r_{an}+1}\mathcal{L}^*_{an}(\chi)(\pi^{r_{an}} - Y^{r_{an}})\mathcal{H} \pmod{\pi^{r_{an}+1}}\end{aligned}$$

which eventually gives us

$$\prod_{i=1}^{r}(U_{\mathfrak{p}_i} - 1)\mathcal{F} \equiv (-1)^{r_{an}+1}\mathcal{L}^*_{an}(\chi)(\pi^{r_{an}} - Y^{r_{an}})\mathcal{H} \pmod{\pi^{r_{an}+1}} \tag{77}$$

We also note that

$$Y^{t+1}\mathcal{H} \equiv \pi^{t+1}\mathcal{E}(\chi,1)\mathcal{W} \equiv 0 \pmod{\pi^{r_{an}+1}} \tag{78}$$

$$Y^{t+1}\mathcal{F} \equiv 0 \pmod{\pi^{r_{an}+1}} \tag{79}$$

where the first equivalence holds due to our assumption on s, t at the beginning of this section.

Also,

$$Y^{t}\mathcal{H} \equiv \pi^{t}\mathcal{E}(\chi,1)\mathcal{W} \equiv \pi^{t}\mathcal{W}\mathcal{H} \pmod{\pi^{r_{an}+1}} \tag{80}$$

which implies

$$(\pi^{t}\mathcal{W} - Y^{t})\mathcal{F} \equiv 0 \pmod{\pi^{r_{an}+1}} \tag{81}$$

The calculations from eq. (69) to eq. (81) gives a surjective map $W_3 \to \phi(\mathbf{T} \otimes E)$. Our objective again is to show that this map in an injection as well. For this it suffices to show that the forms

$$\{\pi^i \mathcal{F}\}_{i=0}^s \cup \{Y^i \mathcal{F}\}_{i=1}^t \cup \left\{ \prod_{j \in J} (U_{\mathfrak{p}_j} - 1)\mathcal{F} \right\}_{\emptyset \neq J \subset R}$$

are linearly independent over E modulo $\pi^{r_{an}+1}$.

Take a linear combination of the forms stated above that sums to zero. We wish to the show that the coefficients appearing in the combination are all zero. Notice that $\mathcal{F}, \pi\mathcal{F}, \ldots, \pi^{s-t}\mathcal{F}$ have different orders of vanishing less than $\pi^{r_{an}}$ and so they must be linearly independent i.e., their coefficients in the combination must vanish. Next, $\pi^{s-t+i}\mathcal{F}$ and $Y^i\mathcal{F}$ have the same orders of vanishing for $i = 1, \ldots, t-1$ and their leading terms are linearly independent. Hence, their coefficients are zero. It remains to show that $\pi^s\mathcal{F}$ and $\left\{\prod_{j\in J}(U_{\mathfrak{p}_j} - 1)\mathcal{F}\right\}_{\emptyset \neq J \subset R}$ are linearly independent over E. But, upto non-zero scalars, these forms are congruent to $\pi^{r_{an}} E_1(1, \chi_J)$ for $\emptyset \neq J \subset R$. We have already seen in Case 1 that such forms are linearly independent. This completes the proof in the second case.

4. General rank, case $3: R' = \emptyset, \nu_1(\mathcal{W}) \neq 0$

We define the $\Lambda_{(1)}$-algebra

$$E[\pi, \epsilon_1, \ldots, \epsilon_r, y]/I_{W_3}$$

where

$$I_{W_3} = \langle \pi^{r_{an}+1}, y^{r_{an}+1}, y(\pi - y), \pi^{r_{an}}\mathcal{W} - y^{r_{an}}(\mathcal{W}+1), \epsilon_i^2, \epsilon_i\pi, \epsilon_i y, \epsilon_1 \cdots \epsilon_r + (-1)^s \mathcal{L}_{an}^*(\chi)(\pi^{r_{an}} - y^{r_{an}})\rangle$$

THEOREM 67. *We have a Λ-algebra homomorphism*

$$\phi : \mathbf{T} \to W_3$$

such that

$$\begin{aligned} T_\mathfrak{l} &\mapsto 1 + \chi\varepsilon(\mathfrak{l}) + \frac{(\chi(\mathfrak{l}) - 1)(1 - \varepsilon(\mathfrak{l}))}{\pi} y && \mathfrak{l} \nmid \mathfrak{n}p, \\ U_\mathfrak{l} &\mapsto 1 + \frac{1 - \varepsilon(\mathfrak{l})}{\pi} y && \mathfrak{l} \mid \mathfrak{n} \text{ or } \mathfrak{l} \in R', \\ U_{\mathfrak{p}_i} &\mapsto 1 + \epsilon_i && R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} \end{aligned}$$

4.1. Proof of theorem 67. In the proof of Case 2, the computations till eq. (77) do not use the assumption on $\mathcal{W}$ and hence, we easily obtain a surjective $\Lambda_{(1)}$-algebra homomorphism

$$W_3 \to \phi(\mathbf{T}) \otimes E$$

Again, we want to show that the above homomorphism is an injection. To show that it suffices to prove that the forms

$$\{\pi^i \mathcal{F}\}_{i=0}^{r_{an}-1} \cup \{Y^i \mathcal{F}\}_{i=1}^{r_{an}} \cup \left\{ \prod_{j \in J} (U_{\mathfrak{p}_j} - 1)\mathcal{F} \right\}_{\emptyset \neq J \subset R}$$

are linearly independent over E modulo $\pi^{r_{an}+1}$. The proof of this is where we use $\nu_1(\mathcal{W}) \neq 0$ crucially. The proof is same as in the previous case, and is given in details in [DKV18, pp. 854-855].

CHAPTER 6

Construction of cohomology class

1. Galois Representations attached to Hida families

Recall the Λ-algebra homomorphism

$$\phi : \mathbf{T} \to W$$

where $W = W_i, i = 1, 2, 3$. If we let $\mathfrak{m}_W$ to be the maximal ideal of W, and $\mathfrak{m} \subseteq \mathbf{T}$ the kernel of the composition

$$\mathbf{T} \xrightarrow{\phi} W \to W/\mathfrak{m}_W \simeq E$$

Let $\mathbf{T}_{(\mathfrak{m})}$ be the localisation of $\mathbf{T}$ at the prime ideal $\mathfrak{m}$. As $\mathbf{T}_{(\mathfrak{m})}$ is Noetherian and reduced, the total ring of fractions of the local ring $\mathbf{T}_{(\mathfrak{m})}$ embeds in L isomorphic to the product of fields

$$L = \prod_{i=1}^{t} L_{\mathcal{H}_i} \tag{82}$$

where $L_{\mathcal{H}_i}$ is a finite extension of $\mathrm{Frac}(\Lambda)$ and corresponds to a cuspidal Hida eigenfamily $\mathcal{H}_i$.

For an integral ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, the normalised Fourier coefficient $c(\mathfrak{a}, \mathcal{H}_i)$ is the image in $L_{\mathcal{H}_i}$ of the Hecke operator $T_{\mathfrak{a}}$. These coefficients generate a Λ-subalgebra of $L_{\mathcal{H}_i}$ which we shall denote by $\Lambda_{\mathcal{H}_i}$. The image of $\mathbf{T}_{(\mathfrak{m})}$ under the injection $\mathbf{T}_{(\mathfrak{m})} \to L$ is the localisation of $\Lambda_{\mathcal{H}_i}$ at a height one prime ideal $\mathfrak{m}_{\mathcal{H}_i}$ lying above $\langle T \rangle \subseteq \Lambda$. The explicit form of the homomorphism ϕ gives us the following equivalences:

$$c(\mathfrak{l}, \mathcal{H}_i) \equiv 1 + \chi(\mathfrak{l}) \pmod{\mathfrak{m}_{\mathcal{H}_i}} \text{ for } \mathfrak{l} \nmid \mathfrak{n}\mathfrak{p} \tag{83}$$

$$c(\mathfrak{l}, \mathcal{H}_i) \equiv 1 \pmod{\mathfrak{m}_{\mathcal{H}_i}} \text{ for } \mathfrak{l} \mid \mathfrak{n}\mathfrak{p} \tag{84}$$

Note that the congruences eq. (83) hold for all $\mathcal{H}_i$ appearing in the product eq. (82). We will use the following result of Hida and Wiles in our construction for cohomology class.

Remark 68 (Notation)**.** *For purposes of notational convenience, I will label $\mathcal{H}_i$ with the index i. So, $\mathcal{H}_i$ will be denoted by i and $L_{\mathcal{H}_i} = L_i, \mathfrak{m}_{\mathcal{H}_i} = \mathfrak{m}_i$ and so on.*

THEOREM 69. *There exists a continuous irreducible Galois representation*

$$\rho_i : G_F \to \mathrm{GL}_2(L_i)$$

where L_i is endowed with the Λ-adic topology (i.e. the topology generated by the maximal ideal $\langle \pi_E, T\rangle$ of Λ, where π_E is the uniformiser of E) satisfying the following conditions:

(1) $\rho_{\mathcal{G}}$ *is unramified outside* $\mathfrak{n}p$,

(2) *for primes* $\mathfrak{l} \nmid \mathfrak{n}p$, *the characteristic polynomial of* $\rho_{\mathcal{H}}(\mathrm{Frob}_{\mathfrak{l}})$ *is*

$$\mathrm{charpoly}(\rho_i(\mathrm{Frob}_{\mathfrak{l}}))(x) = x^2 - c(\mathfrak{l}, \mathcal{H}_i)x + \chi\varepsilon(\mathfrak{l}) \tag{85}$$

(3) *for all* $\mathfrak{p} \mid p$, *the representation has a very specific form. More precisely,*

$$\rho_{\mathcal{H}}|_{G_{\mathfrak{p}}} \sim \begin{pmatrix} \chi\varepsilon\eta_{i,\mathfrak{p}}^{-1} & * \\ 0 & \eta_{i,\mathfrak{p}} \end{pmatrix} \tag{86}$$

where $\eta_{i,\mathfrak{p}} : G_{\mathfrak{p}} \to \Lambda_{\mathcal{H}}^{\times}$ *is unramified and* $\eta_{i,\mathfrak{p}}(\mathrm{rec}(\varpi^{-1})) = c(\mathfrak{p}, \mathcal{H}_i)$. *Here,* $\varpi \in F_{\mathfrak{p}}^{\times}$ *is the uniformiser and* $\mathrm{rec} : F_{\mathfrak{p}}^{\times} \to G_{\mathfrak{p}}^{ab}$ *is the local Artin reciprocity map.*

Notice that locally we have a basis that makes the representation upper triangular as seen in eq. (86). And, globally we have a basis that makes the representation diagonal. Our construction of cohomology class will crucially depend on the interplay between these local and global basis. Mazur calls this the Ribet wrench, and also observes that this method fails if the global and local basis are the same. Thus, we need to find the appropriate basis first that does not agree with the local bases. And, we must also do this simultaneously for the $\mathcal{H}_i$s. To this effect, we will record the following technical lemma which is proven in [DKV18]

Lemma 70. *[[DKV18, Lemma 4.3, pp. 860]] Let* $v_{i,\mathfrak{p}} \in L_i^2$ *be a vector in the representation* ρ_i *that is an eigenvector for* $\rho_i(G_{\mathfrak{p}})$. *Then, there exists a* $\tau \in G_F$ *such that* $\chi(\tau) \neq 1$ *and such that* $v_{i,\mathfrak{p}}$ *is not an eigenvector for* $\rho_i(G_{\mathfrak{p}})$ *for all* $i = 1, \ldots, t$ *and* $\mathfrak{p} \in R$.

Next, let $\overline{\mathbf{T}}$ be the image of $\mathbf{T}$ in $\mathbf{T}_{(\mathfrak{m})}$. The product of the Galois representations ρ_i for $i = 1, \ldots, t$ gives a continuous Galois representation

$$\rho : G_F \to \mathrm{GL}_2(L)$$

satisfying

(1) ρ is unramified outside $\mathfrak{n}p$,

(2) for primes $\mathfrak{l} \nmid \mathfrak{n}p$, the characteristic polynomial of $\rho_{\mathcal{H}}(\mathrm{Frob}_{\mathfrak{l}})$ is

$$\mathrm{charpoly}(\rho(\mathrm{Frob}_{\mathfrak{l}}))(x) = x^2 - \overline{T}_{\mathfrak{l}}x + \chi\varepsilon(\mathfrak{l}) \tag{87}$$

where $\overline{T}_{\mathfrak{l}}$ is the image of $T_{\mathfrak{l}}$ in $\overline{\mathbf{T}}$.

(3) for all $\mathfrak{p} \mid p$, the representation has a very specific form. More precisely,

$$\rho|_{G_{\mathfrak{p}}} \sim \begin{pmatrix} \chi\varepsilon\eta_{\mathfrak{p}}^{-1} & * \\ 0 & \eta_{\mathfrak{p}} \end{pmatrix} \tag{88}$$

where $\eta_{\mathfrak{p}} : G_{\mathfrak{p}} \to \Lambda_{\mathcal{H}}^{\times}$ is unramified and $\eta_{\mathfrak{p}}(\mathrm{rec}(\varpi^{-1})) = \overline{U_{\mathfrak{l}}}$. Here, $\varpi \in F_{\mathfrak{p}}^{\times}$ is the uniformiser and $\mathrm{rec} : F_{\mathfrak{p}}^{\times} \to G_{\mathfrak{p}}^{ab}$ is the local Artin reciprocity map.

Let $\mathbf{T}_{\mathfrak{m}}$ denote the completion of $\mathbf{T}_{(\mathfrak{m})}$ at its maximal ideal. As a consequence of Hensel's lemma, we can find distinct $\lambda_1, \lambda_2 \in \mathbf{T}_{\mathfrak{m}}$ and a basis such that

$$\rho(\tau) = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \tag{89}$$

for the τ obtained in lemma 70.

For $\sigma \in G_F$, let

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \tag{90}$$

Using the eq. (87), $1 + \chi(\mathfrak{l}) \equiv \overline{T_{\mathfrak{l}}} \pmod{\mathfrak{m}}$ and Cebotarev's density theorem (we can apply because $\mathbf{T}$ and $\mathfrak{m} \subseteq \mathbf{T}$ are finitely generated Λ-modules and as ρ is continuous they are closed in the Λ-adic topology) it follows that

$$a(\sigma) + d(\sigma) \in \overline{\mathbf{T}} \subseteq \mathbf{T}_{\mathfrak{m}} \tag{91}$$

and

$$a(\sigma) + d(\sigma) \equiv 1 + \chi(\sigma) \pmod{\mathfrak{m}\overline{\mathbf{T}}} \tag{92}$$

If $\widehat{\mathfrak{m}} = \mathfrak{m}\mathbf{T}_{\mathfrak{m}}$ be the maximal ideal of $\mathbf{T}_{\mathfrak{m}}$. From eq. (89) we have

$$a(\tau) = \lambda_1 \equiv 1 \pmod{\widehat{\mathfrak{m}}} \tag{93}$$

$$d(\tau) = \lambda_2 \equiv \chi(\tau) \pmod{\widehat{\mathfrak{m}}} \tag{94}$$

We also have

$$1 + \chi(\sigma)\chi(\tau) \equiv a(\sigma\tau) + d(\sigma\tau) \pmod{\widehat{\mathfrak{m}}} \tag{95}$$

$$\equiv a(\sigma) + d(\sigma)\chi(\tau) \pmod{\widehat{\mathfrak{m}}} \tag{96}$$

where eq. (95) follows from eq. (92) and eq. (96) follows from eq. (93),eq. (94). From eq. (92) and eq. (96) we can conclude that

$$a(\sigma) \equiv 1 \pmod{\widehat{\mathfrak{m}}} \tag{97}$$

$$d(\sigma) \equiv \chi(\sigma) \pmod{\widehat{\mathfrak{m}}} \tag{98}$$

Next, let B be the $\mathbf{T}_{\mathfrak{m}}$-submodule generated by $b(\sigma)$ where $\sigma \in G_F$. B is in fact a finitely generated $\mathbf{T}_{\mathfrak{m}}$-module. Indeed, suppose B_0 is the $\mathbf{T}_{(\mathfrak{m})}$-submodule generated by $b(\sigma)$ where $\sigma \in G_F$. As ρ is continuous and G_F is compact, B_0 must be compact, hence finitely generated as $\mathbf{T}_{(\mathfrak{m})}$-submodule. And as a result, C is finitely generated as $\mathbf{T}_{\mathfrak{m}}$-submodule. Define the E-vector space

$$\overline{B} := B/\widehat{\mathfrak{m}}B$$

For $\sigma, \sigma' \in G_F$, we have the relation:

$$b(\sigma\sigma') = a(\sigma)b(\sigma') + b(\sigma)d(\sigma') \tag{99}$$

$$= b(\sigma') + \chi(\sigma')b(\sigma) \tag{100}$$

Define the function

$$\kappa(\sigma) := \overline{b}(\sigma)\chi^{-1}(\sigma) \tag{101}$$

where $\overline{b}$ is the image of b in $\overline{B}$. Then, from eq. (99)

$$\kappa(\sigma\sigma') = \chi^{-1}(\sigma)b(\sigma')\chi^{-1}(b') + b(\sigma)\chi^{-1}(\sigma) \tag{102}$$

$$= \sigma \cdot \kappa(\sigma') + \kappa(\sigma) \tag{103}$$

From eq. (102), it is clear that $[\kappa]$ is an element of $H^1(G_F, \overline{B}(\chi^{-1}))$.

This finishes our construction of the cohomology class. In the next section, we will study it in more detail and see what happens in the local bases.

2. Local behaviour of the cohomology class

Recall the cohomology class $[\kappa] \in H^1(G_F, \overline{B}(\chi^{-1}))$ defined via

$$\kappa(\sigma) = \overline{b}(\sigma)\chi^{-1}(b)$$

For each prime $\mathfrak{p} \mid p$, there is a basis for which the matrix $\rho|_{G_{\mathfrak{p}}}$ has the form eq. (88). Let $\begin{pmatrix} A_{\mathfrak{p}} & B_{\mathfrak{p}} \\ C_{\mathfrak{p}} & D_{\mathfrak{p}} \end{pmatrix} \in \mathrm{GL}_2(L_{\mathfrak{m}})$ be the change of basis matrix that transforms the mentioned local basis to the global basis as constructed in eq. (89). More concretely, this means

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} A_{\mathfrak{p}} & B_{\mathfrak{p}} \\ C_{\mathfrak{p}} & D_{\mathfrak{p}} \end{pmatrix} = \begin{pmatrix} A_{\mathfrak{p}} & B_{\mathfrak{p}} \\ C_{\mathfrak{p}} & D_{\mathfrak{p}} \end{pmatrix} \begin{pmatrix} \chi\varepsilon\eta_{\mathfrak{p}}^{-1} & * \\ 0 & \eta_{\mathfrak{p}} \end{pmatrix} \tag{104}$$

for all $\sigma \in G_F$. The first observation is that

Lemma 71. *The elements $A_{\mathfrak{p}}, C_{\mathfrak{p}}$ are invertible.*

Proof. This is because of the choice of τ we have made in lemma 70. □

On comparing the top left entry on both sides, we obtain

$$A_{\mathfrak{p}}a(\sigma) + b(\sigma)C_{\mathfrak{p}} = A_{\mathfrak{p}}\chi\varepsilon\eta_{\mathfrak{p}}^{-1}$$

$$b(\sigma) = \frac{A_{\mathfrak{p}}}{C_{\mathfrak{p}}}\left(\chi\varepsilon\eta_{\mathfrak{p}}^{-1} - a(\sigma)\right) \tag{105}$$

for all $\sigma \in G_F$.

Lemma 72. *The cohomology class* $[\kappa] \in H^1(G_F, \overline{B}(\chi^{-1}))$ *is unramified outside* R.

Proof. [DKV18, Lemma 4.7, pp. 864] □

Lemma 73. *Let* $R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. *We have*

$$B \subseteq \frac{A_{\mathfrak{p}_1}}{C_{\mathfrak{p}_1}}\widehat{\mathfrak{m}} + \cdots + \frac{A_{\mathfrak{p}_r}}{C_{\mathfrak{p}_r}}\widehat{\mathfrak{m}}$$

Proof. [DKV18, Lemma 4.9, pp. 865] □

CHAPTER 7

Gross-Stark regulator computation

1. The homomorphism $\phi_{\mathfrak{m}}$

W is a local Artin ring which is complete with respect to its maximal ideal $\mathfrak{m}_W$ as $\mathfrak{m}_W^{r_{an}+1} = 0$. Hence, the homomorphism $\phi : \mathbf{T} \to W$ extends canonically to a surjective homomorphism

$$\phi_{\mathfrak{m}} : \mathbf{T}_{\mathfrak{m}} \to W$$

In Cases 2 and 3, we define a modified cyclotomic character which Dasgupta-Kakde-Ventullo call the Λ-*adic cyclotomic character in the variable* y:

$$\varepsilon_y : G_F \to W^\times$$

to be sending $\sigma in G_F$ to

$$\varepsilon_y(\sigma) = \sum_{i=0}^{\infty} a_i y^i \tag{106}$$

$$= 1 + \frac{\varepsilon(\sigma) - 1}{\pi} y \tag{107}$$

where a_is are defined as $\sum_{i=0}^{\infty} a_i \pi^i := \varepsilon(\sigma)$, and $a_0 = 1$.

As y is nilpotent, the sum is in fact finite. And, the second relation holds using the relation $\pi y = y^2$ in W_2, W_3.

Define $\varepsilon_{\pi - y}$ similarly and also define two homomorphisms

$$\psi_1, \psi_2 : G_F \to W^\times$$

via

$$\psi_1(\sigma) = \begin{cases} 1 & \text{Case } 1 \\ \varepsilon_y(\sigma) & \text{Case } 2, 3 \end{cases}$$

and

$$\psi_2(\sigma) = \begin{cases} \chi\varepsilon(\sigma) & \text{Case } 1 \\ \chi\varepsilon_{\pi-y} & \text{Case } 2, 3 \end{cases}$$

The main result of this section that will come handy when computing the regulator is

Proposition 74. *We have*

$$\phi_{\mathfrak{m}}(a(\sigma)) = \psi_1(\sigma) \tag{108}$$

$$\phi_{\mathfrak{m}}(d(\sigma)) = \psi_2(\sigma) \tag{109}$$

PROOF. For $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{p}$, we have

$$\phi_{\mathfrak{m}}(T_{\mathfrak{l}}) = \psi_1(\mathrm{Frob}_{\mathfrak{l}}) + \psi_2(\mathrm{Frob}_{\mathfrak{l}}) \tag{110}$$

This implies

$$\phi_{\mathfrak{m}}(a(\sigma) + d(\sigma)) = \psi_1(\sigma) + \psi_2(\sigma) \tag{111}$$

for all $\sigma \in G_F$. Using the relation $\pi y = y^2$ we have

$$\begin{aligned}
(\pi - y)^2 &= \pi^2 - y^2 \\
(\pi - y)^3 &= \pi^3 - y^3 \\
(\pi - y)^2 &= \pi^4 - y^4 \\
&\vdots
\end{aligned}$$

And thus for all $\sigma \in G_F$, we have

$$\begin{aligned}
\varepsilon_{\pi-y}(\sigma) &= \varepsilon(\sigma) + 1 - \varepsilon_y(\sigma) \\
&= \varepsilon(\sigma) - \frac{\varepsilon(\sigma) - 1}{\pi} y \\
\varepsilon_y(\sigma)\varepsilon_{\pi-y}(\sigma) &= \frac{1}{\pi^2}\left[\pi\varepsilon(\sigma) - y(\varepsilon(\sigma) - 1)\right]\left[\pi + y(\varepsilon(\sigma) - 1)\right] \\
&= \frac{1}{\pi^2}\left[\pi^2\varepsilon(\sigma) - y^2(\varepsilon(\sigma) - 1) + y^2\varepsilon(\sigma)(\varepsilon(\sigma) - 1) - y^2(\varepsilon(\sigma) - 1)^2\right] \\
&= \frac{1}{\pi^2}\left[\pi^2\varepsilon(\sigma) - y^2\varepsilon(\sigma) + y^2 + y^2\varepsilon(\sigma)^2 - y^2\varepsilon(\sigma) - y^2\varepsilon(\sigma)^2 - y^2 + 2y^2\varepsilon(\sigma)\right] \\
&= \frac{1}{\pi^2} \cdot \pi^2\varepsilon(\sigma) \\
&= \varepsilon(\sigma) \qquad (112)
\end{aligned}$$

And thus,

$$\psi_1\psi_2 = \chi\varepsilon \tag{113}$$

Using the fact that $\psi_1 \equiv 1 \pmod{\mathfrak{m}_W}$ and $\psi_2 \equiv \chi \pmod{\mathfrak{m}_W}$, and

$$\phi_{\mathfrak{m}}(\mathrm{charpoly}(\rho(\sigma))(x)) = (x - \psi_1(\sigma))(x - \psi_2(\sigma)) \tag{114}$$

which follows from eq. (113), eq. (111), we have

$$\phi_{\mathfrak{m}}(\lambda_1) = \psi_1(\tau) \tag{115}$$

$$\phi_{\mathfrak{m}}(\lambda_2) = \psi_2(\tau) \tag{116}$$

In eq. (111), put $\sigma\tau$ instead of σ to get

$$\phi_{\mathfrak{m}}(a(\sigma))\psi_1(\tau) + \phi_{\mathfrak{m}}(d(\sigma))\psi_2(\tau)$$

$$= \psi_1(\sigma\tau) + \psi_2(\sigma\tau) \tag{117}$$

From eq. (111) and eq. (117), we obtain our conclusion. □

2. Proof of $\mathcal{L}_{an}(\chi) = \mathcal{R}_p(\chi)$ for Cases 1, 2 and 3

This section just follows the paper [DKV18]. If not stated otherwise, $\mathfrak{p}_i$ will be replaced by just the index i, i.e., if there was supposed to be subscript $\mathfrak{p}_i$, we shall just use the subscript i.

Recall that we had $u_1, \ldots, u_r$ an E-basis of U_χ. As a result of **??** and lemma 72, we have

$$\sum_{i=1}^{r} \operatorname{res}_{\mathfrak{p}_i} \kappa(u_j) = 0 \text{ in } \overline{B} \tag{118}$$

for $j = 1, \ldots, r$, and κ is the cohomology class constructed in previous chapter. As $u_j \in U_\chi$, it can be written as

$$u_j = \sum_k y_{jk} \otimes e_{jk} \tag{119}$$

where $y_{jk} \in \mathcal{O}_H[1/p]^\times$ and $e_{jk} \in E$. This is can be done for each j. For each $i = 1, \ldots, r$, let $y_{jk}^{(i)} \in G_{\mathfrak{p}_i}$ such that $\operatorname{rec}_i(y_{jk}) = y_{jk}^{(i)}$. Noting that $\chi(G_{\mathfrak{p}_i}) = 1$, it follows from definition that

$$\operatorname{res}_i \kappa(u_j) = \overline{b}(\sigma_{ij}) \tag{120}$$

where

$$\sigma_{ij} = \sum_k e_{jk} y_{jk}^{(i)} \in E[G_{\mathfrak{p}_i}]$$

Hence, the orthogonality relation in eq. (118) becomes (using 120)

$$\sum_{i=1}^{r} b(\sigma_{ij}) \in \widehat{\mathfrak{m}}B \text{ for each } j = 1, \ldots, r \tag{121}$$

Using eq. (105) we have

$$b(\sigma_{ij}) = \sum_k e_{jk} \cdot \frac{A_i}{C_i} \left(\varepsilon\eta_i^{-1}(y_{jk}^{(i)}) - a(y_{jk}^{(i)}) \right) \tag{122}$$

If we let $\mathbf{I}$ the kernel of the homomorphism $\phi_m : \mathbf{T}_m \to W$, then we have

$$\eta_i^{-1}(y_{jk}^{(i)}) = U_{\mathfrak{p}_i}^{o_i(y_{jk})} \equiv 1 + o_i(y_{jk})(U_{\mathfrak{p}_i} - 1) \pmod{\mathbf{I}}$$

$$\varepsilon(y_{jk}^{(i)}) \equiv 1 + \ell_i(y_{jk})\pi \pmod{\pi^2} \tag{123}$$

$$a(y_{jk}^{(i)}) \equiv 1 + a_i'(y_{jk}) \pmod{\langle \widehat{\mathfrak{m}}^2, \mathbf{I} \rangle} \tag{124}$$

where $a'(y_{jk}) \in \widehat{\mathfrak{m}}$ is an element such that

$$\phi_{\mathfrak{m}}(a_i'(y_{jk})) = \begin{cases} 0 & \text{Case } 1 \\ \ell_i(y_{jk})y & \text{Cases } 2, 3 \end{cases}$$

Next,

$$\varepsilon\eta_i^{-1}(y_{jk}^{(i)}) - a(y_{jk}^{(i)}) \equiv \ell_i(y_{jk})\pi + o_i(y_{jk})(U_{\mathfrak{p}_i} - 1) - a_i'(y_{jk}) \pmod{\langle \widehat{\mathfrak{m}}^2, \mathbf{I}\rangle} \tag{125}$$

Thus, eq. (122) becomes

$$b(\sigma_{ij}) = \sum_k e_{jk} \cdot \frac{A_i}{C_i} \left(\ell_i(y_{jk})\pi + o_i(y_{jk})(U_{\mathfrak{p}_i} - 1) - a_i'(y_{jk}) + m_{ij}\right) \tag{126}$$

for some $m_{ij} \in \langle \widehat{\mathfrak{m}}^2, \mathbf{I}\rangle$. Next, from 73 we have

$$\sum_k e_{jk} \cdot \frac{A_i}{C_i} \left(\ell_i(y_{jk})\pi + o_i(y_{jk})(U_{\mathfrak{p}_i} - 1) - a_i'(y_{jk}) + m_{ij}\right) = 0$$

after changing m_{ij} by elements of $\widehat{\mathfrak{m}}$ is necessary. Consider the matrix

$$\left(\frac{A_i}{C_i} \left(\ell_i(u_j)\pi + o_i(u_j)(U_{\mathfrak{p}_i} - 1) - a_i'(u_j) + m_{ij}\right)\right)_{i,j=1,\ldots,r}$$

The rows of the matrix above sum to zero and hence its determinant must be zero. If we apply the homomorphism $\phi_{\mathfrak{m}}$ to the determinant relation of the above matrix, we get the following equations in the ring W.

$$\det(\ell_i(u_j)\pi + o_i(u_j)\epsilon_i + n_{ij}) = 0 \text{ for Case } 1 \tag{127}$$

$$\det(\ell_i(u_j)(\pi - y) + o_i(u_j)\epsilon_i + n_{ij}) = 0 \text{ for Cases } 2, 3 \tag{128}$$

with $n_{ij} \in \mathfrak{m}_W^2$.

Now, let us finish the proof in Case 1 using the relations in W_1. Notice the following relations modulo $\mathfrak{m}_W^r$:

$$\begin{aligned} 0 &\equiv \det(\ell_i(u_j)\pi + o_i(u_j)\epsilon_i) \pmod{\mathfrak{m}_W^{r+1}} \\ &\equiv \det(\ell_i(u_j))\pi^r + \det(o_i(u_j))\epsilon_1 \cdots \epsilon_r \pmod{\mathfrak{m}_W^{r+1}} \qquad (129) \\ &\equiv \det(\ell_i(u_j))\pi^r + \det(o_i(u_j))(-1)^{r_{an}+1}\mathcal{L}_{an}^*(\chi)\pi^{r_{an}} \pmod{\mathfrak{m}_W^{r+1}} \qquad (130) \end{aligned}$$

Finally, we can break it into two cases:

$r = r_{an}$: Then, $\mathcal{L}_{an}(\chi) = \mathcal{L}_{an}^*(\chi)$ and as $\pi^r \notin \mathfrak{m}_W^{r+1}$, we conclude that

$$\mathcal{L}_{an} = (-1)^r \det(\ell_i(u_j)) / \det(o_i(u_j)) = \mathcal{R}_p(\chi)$$

as we wanted.

$r_{an} > r$: Here, $\pi^{r_{an}} \equiv 0 \pmod{\mathfrak{m}_W^{r+1}}$, and hence eq. (129) becomes $\det(\ell_i(u_j)) = 0$ because of which $\mathcal{R}_p(\chi) = 0$. As $\mathcal{L}_{an}(\chi) = 0$ as well in this case, we again have the equality.

This concludes the proof.

3. Proof of $\mathcal{L}_{an}(\chi^{-1}) = \mathcal{R}_p(\chi^{-1})$ for Case 3

Here, we mostly repeat the arguments from previous chapter and previous sections of this chapter. While constructing the cohomology class, we were looking at the top right entry and hence the $\mathfrak{b}$-cocycle, here instead we will look at the bottom right entry and get the $\mathfrak{c}$-cocyle.

Let C be the finitely generate $\mathbf{T}_\mathfrak{m}$ submodule generated by the elements $c(\sigma)$ for $\sigma \in G_F$. As before, let $\overline{C} = C/\widehat{\mathfrak{m}}C$. For $\sigma, \sigma' \in G_F$ we have the equation

$$c(\sigma\sigma') = c(\sigma)a(\sigma) + d(\sigma)c(\sigma')$$

As a consequence, we can define the cohomology class

$$[\widetilde{\kappa}] \in H^1(G_F, \overline{C}(\chi))$$

as before. When we analyse the local behaviour we get the relation :

$$c(\sigma) = \frac{C_\mathfrak{p}}{A_\mathfrak{p}}\left(\chi\varepsilon\eta_\mathfrak{p}^{-1}(\sigma) - d(\sigma)\right)$$

for $\sigma \in G_\mathfrak{p}$.

It can be shown that

$$C \subseteq \frac{C_{\mathfrak{p}_1}}{A_{\mathfrak{p}_1}}\mathfrak{h} + \cdots + \frac{A_{\mathfrak{p}_r}}{C_{\mathfrak{p}_r}}\mathfrak{h}$$

where $\mathfrak{h} = \phi_\mathfrak{m}^{-1}(yW)$

As $\phi_\mathfrak{m}(d(\sigma)) = \chi\varepsilon_{\pi-y}(\sigma)$ we have

$$\phi_\mathfrak{m}(\chi\varepsilon\eta_i^{-1}(\sigma) - d(\sigma)) = \varepsilon_y - 1 + o_i(\overline{\sigma})\epsilon_i$$

for $\sigma \in G_{\mathfrak{p}_i}$ and $\mathrm{rec}_\mathfrak{p}(\overline{\sigma}) = \sigma$. Like in the previous section, we have the relation

$$\det(\ell_i(u_j)y + o_i(u_j)\epsilon_i + n_{ij}) = 0 \tag{131}$$

with $n_{ij} \in \mathfrak{m}_W\mathfrak{h}$. Arguing as before, we have the congruence

$$0 \equiv \det(\ell_i(u_j))y^r + \det(o_i(u_j))(-1)^{r_{an}+1}\mathcal{L}^*_{an}(\chi)\pi^{r_{an}} \pmod{\mathfrak{m}_W\mathfrak{h}^r} \tag{132}$$

In the ring W_3, there is a relation

$$y^t = \mathcal{W}\pi^t = (-1)^{s-t}\frac{\mathcal{L}^*(\chi)}{\mathcal{L}^*(\chi^{-1})}\pi^s$$

which allows us to rewrite eq. (132) as

$$0 \equiv \det(\ell_i(u_j))y^r + \det(o_i(u_j))(-1)^{t+1}\mathcal{L}^*_{an}(\chi^{-1})y^t \pmod{\mathfrak{m}_W\mathfrak{h}^r} \tag{133}$$

The congruence yields an equality in the one dimensional E-vector space $\mathfrak{h}^r/\mathfrak{m}_W\mathfrak{h}^r$. Again,

$t = r$: $\mathcal{L}^*_{an}(\chi) = \mathcal{L}_{an}(\chi)$ and we eq. (133) becomes

$$0 = \det(\ell_i(u_j)) + \det(o_i(u_j))(-1)^{r+1}\mathcal{L}_{an}(\chi^{-1})$$

$t > r$: $y^t \in \mathfrak{m}_W\mathfrak{h}^r$ and eq. (133) gives $\det(\ell_i(u_j)) = 0$ because of which $\mathcal{R}_p(\chi^{-1}) = 0$. As $\mathcal{L}_{an}(\chi^{-1}) = 0$ as well in this case, we again have the equality.

APPENDIX A

Dedekind Zeta Function

Let k be a number field, S a finite set of places of k containing the infinite places S_∞ of k. Then, define the Dedekind zeta function for $\mathrm{Re}(s) > 1$ by

$$\zeta_k(s) = \zeta_{k,S_\infty}(s) = \prod_{\mathfrak{p}\notin S_\infty} (1 - \mathbb{N}\mathfrak{p}^{-s})^{-1} = \sum_{0\neq\mathfrak{a}\trianglelefteq\mathcal{O}_k} \frac{1}{\mathbb{N}\mathfrak{a}^s} \tag{134}$$

and more generally

$$\zeta_{k,S}(s) = \prod_{\mathfrak{p}\notin S} (1 - \mathbb{N}\mathfrak{p}^{-s})^{-1} = \sum_{\substack{0\neq\mathfrak{a}\trianglelefteq\mathcal{O}_k \\ \gcd(\mathfrak{a},\mathfrak{p})=1\ \forall\ \mathfrak{p}\in S}} \frac{1}{\mathbb{N}\mathfrak{a}^s} \tag{135}$$

The function above can be meromorphically continued to all of $s \in \mathbb{C}$. The functional equation is discussed in Appendix F.

APPENDIX B

Abelian L-functions

References for this section is [Tat84, §1], [Mar77].

Let k be a number field, S a finite set of places of k containing the infinite places S_∞ of k. Let χ be a complex valued function on the ideals of the ring of integers of k. Define the L-function formally by

$$L(s,\chi) = \prod_{\mathfrak{p}\notin S_\infty} (1-\chi(\mathfrak{p})\mathbb{N}\mathfrak{p}^{-s})^{-1} = \sum_{0\neq\mathfrak{a}\trianglelefteq\mathcal{O}_k} \frac{\chi(\mathfrak{a})}{\mathbb{N}\mathfrak{a}^s} \tag{136}$$

If χ satisfies the asymptotic condition $\chi(\mathfrak{a}) = O(\mathbb{N}\mathfrak{a}^\sigma)$ for $\sigma \in \mathbb{R}$, then $L(s,\chi)$ converges for $\mathrm{Re}(s) > 1+\sigma$.

For example, when $k = \mathbb{Q}$, we have the Dirichlet characters $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times$ for $f \in \mathbb{Z}_{\geq 2}$. The character can be extended to all of $\mathbb{Z}$ by letting $\chi(a) = 0$ if $\gcd(a,f) \neq 1$. For a general k, fix an integral ideal $\mathfrak{f}$ of k and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_\mathfrak{f}^\times \longrightarrow k_\mathfrak{f}^\times \longrightarrow I_\mathfrak{f} \longrightarrow C_\mathfrak{f} \longrightarrow 0$$

where

$$\begin{aligned} \mathcal{O}_\mathfrak{f}^\times &= \{x \in \mathcal{O}_k^\times : x \equiv 1 \mod \mathfrak{f}\} \\ k_\mathfrak{f}^\times &= \{x \in k^\times : x \equiv 1 \mod \mathfrak{f}\} \\ I_\mathfrak{f} &= \{\mathfrak{a} \in I : \mathfrak{a} \equiv 1 \mod \mathfrak{f}\} \end{aligned}$$

and $C_\mathfrak{f}$ the quotient of $I_\mathfrak{f}$ by the principal ideals generated by elements of $k_\mathfrak{f}^\times$. We want to get a character of $I_\mathfrak{f}$ through a character of $C_\mathfrak{f}$

For $k = \mathbb{Q}$, we have $C_f = (\mathbb{Z}/f\mathbb{Z})^\times/\{\pm 1\}$ which does not really correspond to the Dirichlet characters we started with. We thus have to take into consideration the question of sign: if T is a set of real places of k, we denote by $k_{\mathfrak{f},T}^\times$ (resp. $\mathcal{O}_{\mathfrak{f},T}^\times$) the elements of $k_\mathfrak{f}^\times$(resp. $\mathcal{O}_\mathfrak{f}^\times$) that are positive for all places of T. Let $C_{\mathfrak{f},T}$ denote the quotient of $I_\mathfrak{f}$ by the image of $k_{\mathfrak{f},T}^\times$. This is a finite group. To summarise, we have the following commutative diagram whose rows and columns are exact:

$$\begin{tikzcd}
 & & & & 0 & \\
0 \arrow[r] & \mathcal{O}_{\mathsf{T}}^{\times} \arrow[r] & \mathsf{k}_{\mathsf{T}}^{\times} \arrow[r] & \mathrm{I} \arrow[r] & \mathsf{C}_{\mathsf{T}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathfrak{f},\mathsf{T}}^{\times} \arrow[r] \arrow[u] & \mathsf{k}_{\mathfrak{f},\mathsf{T}}^{\times} \arrow[r] \arrow[u] & \mathrm{I}_{\mathfrak{f}} \arrow[r] \arrow[u] & \mathsf{C}_{\mathfrak{f},\mathsf{T}} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & (\mathcal{O}/\mathfrak{f})^{\times}/\mathrm{Im}(\mathcal{O}_{\mathsf{T}}^{\times}) \arrow[u] & \\
 & & & & 0 \arrow[u] &
\end{tikzcd}$$

A homomorphism $\chi : \mathsf{C}_{\mathfrak{f},\mathsf{T}} \to \mathbb{C}^{\times}$ is seen as a function on I by letting $\chi(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not coprime to $\mathfrak{f}$. We thus have

$$\mathsf{L}(s,\chi) := \prod_{\mathfrak{p} \nmid \mathfrak{f}} (1 - \chi(\mathfrak{p})\mathbb{N}\mathfrak{p}^{-s})^{-1}$$

The above product converges for $\mathrm{Re}(s) > 1$.

We say that $\chi : \mathcal{C}_{\mathsf{f}_{\mathsf{T}}} \to \mathbb{C}^{\times}$ is *primitive* (where f_{T} is the conductor of χ), if for all $\mathsf{f}'|\mathsf{f}$ and $\mathsf{T}' \subseteq \mathsf{T}$, there exists a χ' such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}_{\mathsf{f}_{\mathsf{T}}} \arrow[r] \arrow[dr, "\chi"'] & \mathcal{C}_{\mathsf{f}'_{\mathsf{T}'}} \arrow[d, "\chi'"] \\
 & \mathbb{C}^{\times}
\end{tikzcd}$$

implying $\mathsf{f}' = \mathsf{f}, \mathsf{T}' = \mathsf{T}$. By abuse of language, from now on we say $\mathsf{L}(s,\chi)$ is primitive if χ is. Consider a function $\mathsf{L}(s,\chi)$ non-primitive if it removes a few Euler factors.

We know how to analytically continue $\mathsf{L}(s,\chi)$ to the entire complex plane with a functional equation, cf. Appendix F. If $\chi = \mathbf{1}$, $\mathsf{L}(s,\chi)$ is equal to ζ_{K} or one of $\zeta_{\mathsf{k},\mathsf{S}}$ depending on whether $\mathsf{f} = 1$ or not. If $\chi \neq \mathbf{1}$, we know that $\mathsf{L}(s,\chi)$ is holomorphic and $\mathsf{L}(1,\chi) \neq 0$.

<u>In terms of ideles</u> The χs constructed correspond to continuous homomorphisms $\mathbb{A}_{\mathsf{k}}^{\times} \to \mathsf{S}^1$ of finite order and trivial on principal ideals of $\mathsf{k}^{\times}$. In effect, an idele $(x_v) \in \mathbb{A}_{\mathsf{k}}^{\times}$ corresponds to an ideal $\mathfrak{I}_{\mathsf{f}}$ generated by the components $x_{\mathfrak{p}}$ for $\mathfrak{p} \mid \mathsf{f}$. We are not going to, in these notes, concern ourselves with the more general quasi-characters of $\mathbb{A}_{\mathsf{k}}^{\times}$.

The theory of ray class fields establishes, for all pair (f,T) as before, the existence of an unique abelian extension $K_{\mathsf{f},\mathsf{T}}$ of k- namely ray class field f - such that the following three conditions are satisfied:

(1) A prime ideal $\mathfrak{p}$ of k ramifies in $K_{\mathsf{f},\mathsf{T}}$ if and only if $\mathfrak{p} \mid \mathsf{f}$.

Notation: If K/k is an abelian extension with a finite Galois group G, $\mathfrak{p}$ a place of k that does not ramify in K/k and $\mathfrak{P}$ a place of K that divides $\mathfrak{p}$, then we note that $\left(\frac{\mathfrak{p}}{K/k}\right)$ is an unique element of $G_{\mathfrak{P}} \subseteq G$ (see below) whose reduction modulo $\mathfrak{P}$ is the automorphism $x \mapsto x^{N\mathfrak{p}}$ on the residue field of $\mathfrak{P}$. As G is abelian, the above depends only on $\mathfrak{p}$.

(2) The map $\mathfrak{p} \mapsto \left(\frac{\mathfrak{p}}{K_{\mathsf{f},\mathsf{T}}/k}\right)$ induces an isomorphism-namely the Artin reciprocity :

$$\psi_{\mathsf{f}} : \mathcal{C}_{\mathsf{f},\mathsf{T}} \xrightarrow{\sim} \mathrm{Gal}(K_{\mathsf{f},\mathsf{T}}/k)$$

(3) The norm $N_{K_{\mathsf{f},\mathsf{T}}/k}\mathfrak{a}$ of each ideal $\mathfrak{a} \neq 0$ from $K_{\mathsf{f},\mathsf{T}}$ prime to f is a principal ideal generated by an element of $k^{\times}_{\mathsf{f},\mathsf{T}}$.

Moreover, for each finite abelian extension K/k, the Galois group G, there exists a pair (f,T) chosen minimally (called the conductor of) K/k such that

(1) $K \subseteq K_{\mathsf{f},\mathsf{T}}$;
(2) The surjection $\psi_{K/k} : \mathcal{C}_{\mathsf{f},\mathsf{T}} \xrightarrow{\psi_{\mathsf{f}}} \mathrm{Gal}(K_{\mathsf{f},\mathsf{T}}/k) \to G$ is induced from the map $\mathfrak{p} \mapsto \left(\frac{\mathfrak{p}}{K/k}\right)$;
(3) The kernel $\ker \psi_{K/k}$ forms the class of representatives of the norms of the ideals of K.

By $\widehat{G}$ we denote the characters (of dimension 1) of the group G. Thanks to $\psi_{K/k}$, the elements of $\widehat{G}$ can be interpreted as a character of the type envisaged in earlier section. The conductor of $\chi \in \widehat{G}$ is then that of the fixed field of $\ker \chi \subseteq G$. By writing primitive functions everywhere, we prove the following decomposition ([see CF10, p. 217]; [Wei95, pp. XIII–10]):

$$\zeta_K(s) = \prod_{\chi \in \widehat{G}} L(s,\chi) = \zeta_K(s) \prod_{\chi \neq \mathbf{1}} L(s,\chi) \tag{137}$$

APPENDIX C

Linear representations of finite groups

The reference for this section is [Ser77]

Suppose G is a group of finite order g and E a field of characteristic 0. An E-linear representation of G is a homomorphism $\rho : G \to GL(V)$, for a vector space V over E. This amounts to providing V with an $E[G]$-module structure. We can therefore simply talk about the representation V of G.

The character of the representation ρ is a function $\chi = \chi_\rho : G \to E$, such that the trace equals that of action of the automorphism $\rho(x)$ $(x \in G)$ on E. This is a class function (i.e., $\chi(xyx^{-1}) = \chi(y)\ \forall\ x, y \in G$) with $\chi(1) = \dim V$. It takes its values on a cyclotomic extension of $\mathbb{Q}$ contained in E. We denote by $a \mapsto a^*$ the automorphism of the cyclotomic extension of $\mathbb{Q}$ induced by the substitution $\zeta \mapsto \zeta^{-1}$ of roots of unity. For $E \subseteq \mathbb{C}$, we find that $a^* = \bar{a}$ (complex conjugation). Likewise, we write χ^* (or $\bar{\chi}$, if $E \subseteq \mathbb{C}$) for the character obtained by conjugating the values of χ. Two representations of G are isomorphic if and only if they have the same character. This follows from the orthogonality relations between irreducible characters of G (= characters of representations with no proper G-stable subspace), relative to the following scalar product :

$$\langle \chi_1, \chi_2 \rangle_G = \frac{1}{g} \sum_{\sigma \in G} \chi_1(\sigma)\chi_2^*(\sigma) = \frac{1}{g} \sum_{\sigma \in G} \chi_1(\sigma)\chi_2(\sigma^{-1})$$

We note that $\mathbf{1}_G : G \to E$ is just the trivial character corresponding to the dimension of dimension 1. A *virtual character* of G in E is a combination of $\mathbb{Z}$-linear characters of G attached to the representations of G in E.

Properties of $\langle \cdot, \cdot \rangle_G$

Here, the arguments of the scalar product will be that of virtual characters.

(1) $\langle \chi_1, \chi_2 \rangle_G \in \mathbb{Z}$
(2) $\langle \chi_1 + \chi_2, \chi_3 \rangle_G = \langle \chi_1, \chi_3 \rangle_G + \langle \chi_2, \chi_3 \rangle_G$
$\langle \chi_1, \chi_2 \rangle_G = \langle \chi_2, \chi_1 \rangle_G = \langle \chi_1 \chi_2^*, \mathbf{1}_G \rangle_G$
(3) **Frobenius Reciprocity**

Suppose H is a subgroup of G of order h; ψ a virtual character of H and χ a virtual character of G. So,

$$\langle \psi, \chi|_H \rangle_H = \langle \mathrm{Ind}_H^G \psi, \chi \rangle_G$$

Here, for $\sigma \in G$, $\mathrm{Ind}_H^G \psi(\sigma) = \dfrac{1}{h} \displaystyle\sum_{\tau \in G, \tau^{-1}\sigma\tau \in H} \psi(\tau^{-1}\sigma\tau)$

It is the function induced on G by ψ. If ψ is the character of the representation $H \to GL(W)$, then $\mathrm{Ind}\,\psi$ is that of the induced representation $E[G] \otimes_{E[H]} W$ of G.

Given an $E[G]$-module V and a subgroup H of G, we let $V^H = \{x \in V : \sigma x = x \ \forall \sigma \in H\}$. If W is also an $E[G]$-module, the action of G on $V \otimes_E W$ is given by $\sigma(x \otimes y) = \sigma x \otimes \sigma y$ and on $\mathrm{Hom}_E(V, W)$ by $(\sigma f)(x) = \sigma(f(\sigma^{-1}x))$ so that $\mathrm{Hom}_{E[G]}(V, W) = \mathrm{Hom}_E(V, W)^G$. If V (resp. W) is a representation of G over E of the character χ (resp. ψ), then $\chi\psi$ is a character of $V \otimes_E W$ while the one of $\mathrm{Hom}_E(V, W)$ is $\chi^*\psi$. In fact, we have $V \otimes_E W \simeq \mathrm{Hom}_E(V^*, W)$ where $V^* = \mathrm{Hom}_E(V, E)$ is the dual of V. According to what we have said, the conjugate character χ^* of the character χ is attached to the action of G on V.

From now onwards, the group G given will be the Galois group of the finite extension K/k of global fields. We will always assume the action of G is on the left. However, we sometimes write a^σ instead of σa (for $\sigma \in G, a \in K$). In these cases, the reader should be accustomed to the formula $a^{\sigma\tau} = (a^{tau})^\sigma$ for $\sigma, \tau \in G, a \in K$.

If w is a place of K, G_w is used to denote the decomposition group of w with respect to K/k, i.e., $G_w = \{\sigma \in G : \sigma w = w\}$. If w is non-archimedean, I_w is used to denote the inertia group of w, formed by the elements of G_w that induces trivial automorphism on the residual extension. So, if v is the restriction of w in k, the Galois group of the residual extension of w/v is identified with G_w/I_w and one notes that $\sigma_w \in G_w/\mathrm{I}_w$ is the Frobenius automorphism (elevating to a power of Nv on the residue field of w). σ_w generates G_w/I_w.

If w is archimedean, we sometimes write σ_w for the unique generator of G_w. In fact, in the case G_w is of order 2 or 1 depending on whether w is complex extension of a real place or not.

APPENDIX D

Definition and properties of Artin L-functions

Suppose K/k is a finite Galois extension of number fields with Galois group G. Let $\chi : G \to \mathbb{C}$ be a character of a complex representation $G \to GL(V)$. With the notations as in the previous section, for each place $\mathfrak{P}$ of K, the element $\sigma_{\mathfrak{P}} \in G_{\mathfrak{P}}/I_{\mathfrak{P}}$ acts on $V^{I_{\mathfrak{P}}}$. Note that, for $\mathrm{Re}(s) > 1$,

$$L(s, V) = \prod_{\mathfrak{p}} \det(1 - \sigma_{\mathfrak{P}} N\mathfrak{p}^{-s} | V^{I_{\mathfrak{P}}})^{-1} \tag{138}$$

where $\mathfrak{p}$ denotes a finite place of k and for each $\mathfrak{p}$, $\mathfrak{P}$ is a place of K dividing $\mathfrak{p}$ (arbitrarily chosen). The $\sigma_{\mathfrak{P}}$ given are conjugates of each other, thus the value of the "characteristic polynomial" of $\sigma_{\mathfrak{P}}$ appearing as a member in the product is independent of the choice of $\mathfrak{P}$.

The same argument shows that $L(s, V)$ remains unchanged if change V by an isomorphic representation. We can therefore write $L(s, \chi)$ without ambiguity instead of $L(s, V)$. In fact, here is an explicit formula due to Artin which depends only on χ :

$$\log L(s, \chi) = \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \frac{\chi(\sigma_{\mathfrak{P}}^n)}{n \cdot N\mathfrak{p}^{ns}} \tag{139}$$

where $\chi(\sigma_{\mathfrak{P}}^n) = \dfrac{1}{|I_{\mathfrak{P}}|} \sum_{\tau \in \sigma_{\mathfrak{P}}^n} \chi(\tau)$

Formal properties:

Once we have shown analytic continuation of $L(s, \chi)$, the following properties become valid for all $s \in \mathbb{C}$.

(1) **Additivity**

$$L(s, \chi_1 + \chi_2) = L(s, \chi_1) + L(s, \chi_2)$$

(2) **Induction**

$$\begin{tikzcd} K \arrow[d, dash, "H"'] \\ k' \arrow[d, dash] \\ k \end{tikzcd}$$

For a subgroup H of G and a character χ of H, denote by $\mathrm{Ind}\,\chi$ the character of G induced from χ. So,

$$L(s, \mathrm{Ind}_H^G \chi) = L(s, \chi)$$

(3) **Inflation**

$$\begin{tikzcd} K \arrow[d, dash, "H"'] \\ k' \arrow[d, dash, "G'"'] \\ k \end{tikzcd}$$

For a quotient $G' = G/H$ where H is a distinct subgroup of G and χ a character of G', denote by $\mathrm{Infl}\chi$ the character $G \to G/H \xrightarrow{\chi} \mathbb{C}$. So,

$$L(s, \mathrm{Infl}_{G/H}^G \chi) = L(s, \chi)$$

(4) If $\chi(1) = 1$, that is to say that V is of dimension 1, the homomorphism $\chi : G \to \mathbb{C}^\times$ factorises through the abelianisation G^{ab} of G.

APPENDIX E

A theorem of Brauer and Artin's conjecture

A character of G is termed *monomial* if it is induced by a character of degree 1 of a subgroup of G. The theorem of Brauer affirms that all characters of G are integral linear combination of irreducible monomial characters.

Thanks to our discussion in last section, we can deduce that each Artin L-function can be written in the form

$$\prod_i L(s, \psi_i)^{n_i}$$

with $n_i \in \mathbb{Z}$ and ψ_i is a character of degree $\psi_i(1) = 1$ of a suitable subgroup H_i of G. On applying the induction property, we can pass to a quotient H_i of $\ker \psi_i$, so that ψ_i becomes a character of cyclic group.

Let χ be a character of a complex representation of G. One cannot always impose on the integers n_i to be positive. Nevertheless, this decomposition tells us that $L(s, \chi)$ has an analytic continuation to a meromorphic function defined on the entire complex plane.

The conjecture due to Artin says that $L(s, \chi)$ is an entire function, if χ does not contain the trivial character $\mathbf{1}_G$ ([Mar77, pp. I–5]).

APPENDIX F

Functional equation

The main references for this section is [Wei95][God95a][God95b]

Let χ be a character of a complex representation of $G = \mathrm{Gal}(K/k)$.

To begin, complete $L(s,\chi)$ with the gamma factors corresponding to the infinite places of k. Let

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$$
$$\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2\cdot(2\pi)^{-s}\Gamma(s)$$

For each infinite place v of k, choose a place w of K lying above v. If G_w has order 2 (cf. 1.4.4), let χ_- be a non-trivial character. In any case, put $\chi_+ = \mathbb{1}_{G_w}$ and write

$$\chi|_{G_w} = n_+(w)\chi_+ + n_-(w)\chi_-$$

So we have $n_+(w) = \dim V$ and $n_-(w) = \mathrm{Codim}\, V^{G_w}$. Using this decomposition, the local factor L_V does not depend on our choice of w, and is defined by the additivity from the formulas : $\begin{cases} L_V(s,\chi_+) = \Gamma_{\mathbb{C}}(s) & \text{if } V \text{ is complex} \\ L_V(s,\chi_+) = \Gamma_{\mathbb{R}}(s) & \text{if } V \text{ is real} \\ L_V(s,\chi_-) = \Gamma_{\mathbb{R}}(s+1) & \text{if } V \text{ is real} \end{cases}$

If r_2 is the number of complex places of k, we set

$$\begin{aligned} a_1 &= a_1(\chi) = \textstyle\sum_{v \text{ real}} \dim V^{G_w} \\ a_2 &= a_2(\chi) = \textstyle\sum_{v|\infty} \mathrm{Codim}\, V^{G_w} = \sum_{v \text{ real}} \mathrm{Codim}\, V^{G_w} \\ n &= [k:\mathbb{Q}] = \tfrac{1}{\chi(1)}(a_1(\chi) + a_2(\chi) + 2r_2\chi(1)) \end{aligned}$$

More explicitly,

$$\prod_{v|\infty} L_V(s,\chi) = 2^{r_2\chi(1)(1-s)}\mathfrak{p}^{-\frac{a_2}{2}-\frac{s}{2}n\chi(1)}\Gamma(s)^{r_2\chi(1)}\Gamma(s/2)^{a_1}\Gamma\left(\frac{s+1}{2}\right)^{a_2} \tag{140}$$

Note that we have $a_i(\chi) = a_i(\overline{\chi})$ for $i = 1, 2, \ldots$

If $\mathfrak{p}$ is a finite place of k, choose a place $\mathfrak{P}$ of K such that $\mathfrak{P}|\mathfrak{p}$. If $I_{\mathfrak{P}} = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ be the sequence of ramification groups of $\mathfrak{P}/\mathfrak{p}$ ([Ser79, ch. IV]). We denote by g_i the cardinality of G_i and put

$$f(\chi,\mathfrak{p}) = \sum_{i=0}^{\infty} \frac{g_i}{g_0}\,\mathrm{Codim}\, V^{G_i} \tag{141}$$

This number does not depend on the choice of $\mathfrak{P}$ and we can show that it is a rational integer ([Ser79, see VI-2]). As we trivially have $f(\chi,\mathfrak{p}) = 0$ if $\mathfrak{p}$ does not ramify in K/k, we can define the Artin conductor of χ by

$$f(\chi) = \prod_{\mathfrak{p}} \mathfrak{p}^{f(\chi,\mathfrak{p})} \tag{142}$$

where $\mathfrak{p}$ denotes all the finite places (prime ideals) of k

We put, with the notations as before:

$$\Lambda(s,\chi) = \{|d_k|^{\chi(1)} Nf(\chi)\}^{s/2} \prod_{v|\infty} L_V(s,\chi)L(s,\chi) \tag{143}$$

where $|d_k| \in \mathbb{Q}$ is the value of the absolute discriminant of k over $\mathbb{Q}$; $Nf(\chi) > 0$ the absolute norm of $f(\chi)$; and for a real positive α and $z \in \mathbb{C}$, we put (here and then) $\alpha^z = \exp(z \log \alpha)$ with $\log \alpha \in \mathbb{R}$.
So, the functional equation of $L(s,\chi)$ can be written as

$$\Lambda(1-s,\chi) = W(\chi)\Lambda(s,\overline{\chi}) \tag{144}$$

with a constant $W(\chi) \in \mathbb{C}^\times$ of modulus 1.

The constant $W(\chi)$-named "Artin's Wurzelzahl" is written as

$$W(\chi) = W_\infty(\chi)\tau(\overline{\chi})(Nf(\chi))^{-1/2} \tag{145}$$

where $W_\infty(\chi) = \prod_{v|\infty} i^{\mathrm{Codim}\, V^{G_w}} = i^{-a_2(\chi)}$ and the complex constants $\tau(\overline{\chi})$ are characterised by the following formalism :

(1) $\tau(\chi_1 + \chi_2) = \tau(\chi_1)\tau(\chi_2)$
(2) $\tau(\mathrm{Ind}_H^G(\chi)) = \tau(\chi)\left((N_{k/\mathbb{Q}}\mathcal{D}(k'/k))^{1/2} i^{m(k'/k)}\right)^{\chi(1)}$

$$\begin{array}{c} K \\ {\scriptstyle H}\Big| \\ k' \\ \Big| \\ k \end{array}$$

where $\mathcal{D}(k'/k)$ is the discriminant ideal of k' on k and $m(k'/k) = \#\{v' : v'|\infty : v' \text{ is a place of } k', G_{v'}(k'/k) \neq \{1\}\}$

(3) If χ is of dimension 1, we interpret accordingly as a Dirichlet character of k, so $\tau(\chi)$ is a Gauss sum involved in the functional equation of the abelian L-function (see [MaD], II-2 for the explicit local formulas).

Note that ([Mar77]) we have $W_\infty(\overline{\chi}) = W_\infty(\chi)$ and $f(\overline{\chi}) = f(\chi)$. Finally, we will rewrite the explicit functional equation by using the following identity ([Mar77,

p. 49]):

$$W(\chi) = \frac{Nf(\chi)^{1/2}}{\tau(\chi)W_\infty(\chi)} \tag{146}$$

The sign of the discriminant $d_k \in \mathbb{Q}$ is $(-1)^{r_2}$. We put

$$\sqrt{d_k} = i^{r_2}|d_k|^{1/2} \in \mathbb{C}$$

With all the notations, here is a explicit version of the functional equation :

$$L(1-s,\chi) = \begin{cases} 2^{r_2\chi(1)} \frac{i^{(a_1+r_2\chi(1))}}{\tau(\chi)\sqrt{d_k}^{\chi(1)}} \pi^{1/2n\chi(1)} \left(\frac{\Gamma(s)}{\Gamma(1-s)}\right)^{r_2\chi(1)} \left(\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\right)^{a_1} \\ \left(\frac{\Gamma((1+s)/2)}{\Gamma((2-s)/2)}\right)^{a_2} B^s L(s,\overline{\chi}) \end{cases} \tag{147}$$

where B is a non-zero positive real.

Let us write $c(\overline{\chi})$ (resp. $c_1(\chi)$) for the first non-zero coefficient in the Laurent series expansion of $L(s,\overline{\chi})$ (resp. $L(s,\chi)$) at $s=0$ (resp. $s=1$) and let $r_1(\chi)$ be the multiplicity of $L(s,\chi)$ at $s=1$. Letting $s \to 0$ in equation for $L(1-s,\chi)$, we can finally obtain (recall that $\Gamma(1/2) = \pi^{1/2}$ and that Γ has a simple pole with residue 1 at $s=0$):

$$\frac{c_1(\chi)}{c(\overline{\chi})} = (-1)^{r_1} 2^{r_2\chi(1)+a_1(\chi)} \frac{(\pi i)^{a_2(\chi)+r_2\chi(1)}}{\tau(\chi)\sqrt{d_k}^{\chi(1)}} \tag{148}$$

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