Zero-Sum Problems in Finite Abelian Groups

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Abstract

Zero sum problems in finite abelian groups are studied using some constants like the Davenport constant $\mathcal{D}(G)$, Erdös-Ginzberg-Ziv constant $s(G)$, and the η-invariant. A major portion of the report is concerning results for rank-2 groups and some discussion on certain rank-3 and higher rank groups.

Contents

The entire text is concerned with only finite abelian groups. So in case it is not clear, safely assume that 'G' is a finite abelian group

1 Introduction

Zero-sum problems are problems relating to the structure of finite Abelian Groups.

1.1 Problem statement

Question 1.1.
Let $(G,+)$ be a finite Abelian Group. Then we ask if given a finite collection of elements say $g_1, g_2, \ldots g_n$ does there exist a certain sub-collection whose element sums to 0.

For the case of Infinite Groups we need to modify the statement a little bit. But, I will not be studying this.

1.2 General observations

Definition.

- If we allow repetitions of g_i we simply call the collection sequences and subsequences. Note that sequences are also called "multisets".
- If we do not allow repetitions of g_i we call the collection sets and subsets. Note that they are also called "square-free sequences".

Lemma 1.2.
Consider a $(G,+)$. Then for sufficiently large n, the sequence $g_1, g_2, \ldots g_n$ certainly contains a subsequence $g_{j_1}, g_{j_2}, \ldots g_{j_k}$ whose entries sum to zero.

Proof. Assume $n = |G|^2$. Then there is an element $g \in (G,+)$ which appears n times and so we can simply take the subcollection to be n copies of that g. Since, $g|G| = 0 \; \forall \; g \in G$ we have proved our claim. □

- It is easy to see that if the sequence is too big then every sequence $g_1, g_2, \ldots g_n$ will have a subsequence whose elements sum to zero.
- What we usually do is put some kind of conditions on the length of the sequence (or cardinality of the set) and check if that is sufficient to get the desired subsequence. That is to say we want Ramsey-like results.
- We want to find a best possible n given $(G,+)$.

Remark 1.3.
- We see that n in the above is quite large and we would like to decrease it, the least such k for which the property is satisfied is known as the "Davenport constant" denoted by $\mathcal{D}(G)$.
- The similar concept can also be expressed in terms of set and the smallest such k in that case is called the "Olson constant" and denoted by $\mathsf{Ol}(G)$.

1.3 Ramsey-like Problems

Given any large structure we would like to study the unavoidable regularity in these large structures. This is the essence of Ramsey theory. We want to know if given a large collection can we partition it into finite number of classes such that there is always one such class which contains all the elements of some regularity condition. In addition to this we would also like to know how big can these "large" structures be and also try to put find some minimum condition on the smaller classes to satisfy the regularity condition.

1.4 Structure of Finite Abelian Groups

The Finite Abelian Groups have a well-defined structure that makes them easy to study. We have listed that below.

Theorem 1.4 (Fundamental Theorem of Finite Abelian Groups)**.**
Consider $(G,+)$ to be a Finite Abelian group. Then there exists integers $n_1, n_2, \dots n_r$ such that $1 < n_1|n_2|\dots|n_r$ and

$$(G,+) \cong (\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_r\mathbb{Z}, +)$$

Some notation needs to fixed now.

Definition.
- The integer r in the above theorem is called the rank of the group and denoted by $r(G)$.
- The integer n_r in the above theorem is called the exponent of the group and denoted by $exp(G)$.

Remark 1.5.
We note that $g\exp(G) = 0\ \forall\ g \in G$ and $exp(G)$ is the smallest such integer with such a property.

Theorem 1.6.
Consider $(G,+)$ to be a Finite Abelian group. Then there are prime powers $q_1, q_2, \ldots q_s$ such that

$$(G,+) \cong (\mathbb{Z}/q_1\mathbb{Z} \oplus \mathbb{Z}/q_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/q_s\mathbb{Z}, +)$$

Here, q_is are uniquely determined upto ordering.

Note 1.7.
We have two notions of rank, so we try to find relation between $r(G)$ and denoted by $r^*(G)$

We now formulate the structure theorem in a way that might be helpful later on.

Theorem 1.8 (Reformulation of FToFAG)**.**
Consider $(G,+)$ to be a Finite Abelian group. Then there exists integers $e_1, e_2, \ldots e_r$ such that $1 < ord(e_1)|ord(e_2)|\ldots|ord(e_r)$ and

$$(G,+) \cong < e_1 > \oplus < e_2 > \oplus \ldots \oplus < e_r >$$

Moreover the orders are uniquely obtained.

Definition.
The set of elements $e_1, e_2, \ldots e_r$ is called the basis of G.

We also try to define a notion of independence of elements in the following way

Definition.
A family $f_1, f_2, \ldots f_s$ over G is said to be independent if given

$$\sum_i^s a_i f_i = 0$$

where $a_i \in \mathbb{Z}$ implies $a_i f_i = 0 \ \forall \ i$.

Generally, any set of independent generators is called a basis.

Remark 1.9.
- The cardinality of the basis set is not unique in general.
  - $r(G)$ is the smallest size of the basis.
  - $r^*(G)$ is the largest size of the basis.
- We have $r(G) = r^*(G)$ iff $\exp(G)$ is a prime power and we call G a p-group.

1.5 Constants of Interest

Definition (of zero-sum constant)**.**
Let $(G,+)$ be a finite abelian group. Let $I \subset \mathbb{N} \backslash \{0\}$. Let $s_I(G) \in \mathbb{N} \cup \{\infty\}$ be the smallest integer such that each sequence $g_1, g_2, \dots g_k$ with $k \geq s_I(G)$ has a subsequence with sum of elements equal to 0 and length in I.

Lemma 1.10.
$s_I(G)$ is finite iff I contains a multiple of $\exp(G)$.

Proof. □

Note 1.11.

1. $s_{\mathbb{N}\backslash\{0\}}(G) = \mathcal{D}(G)$ called Davenport's constant.
2. $s_{\exp(G)}(G) = s(G)$ called the Erdös - Ginzberg - Ziv constant.
3. $s_{|G|}(G) = zs(G)$ called the zero-sum constant.
4. $s_{1,\dots,\exp(G)}(G) = \eta(G)$ called the η- invariant.
5. $s_{\exp(G)\mathbb{N}}(G) = s_o(G)$

2 Structure Theorem of Finite Abelian Groups

Before stating the structure theorem I want to establish a a proposition which will be used extensively.

Proposition 2.1 (Recognition theorem of direct products)**.**
Let G be a group with subgroups H, K that satisfy the following properties:

1. H and K are normal in G.
2. $H \cap K = \{e\}$

Then we have $HK \cong H \times K$

Proof. Let us first prove that the elements in HK commute. Let $h \in H$ and $k \in K$ and H is normal in K. Then we can say that

$$hkh^{-1} = k_1 \text{for some } k_1 \in K$$
$$hkh^{-1}k^{-1} \in K$$

And

$$kh^{-1}k^{-1} = h_1 \text{ for some } h_1 \in H$$
$$hkh^{-1}k^{-1} \in H$$

But from condition 2 we see that $hkh^{-1}k^{-1} = e \Rightarrow hk = kh$. Also to note is that since H, K. Now that we have shown this we define a homomorphism

$$\phi : HK \to H \times K$$

such that $hk \mapsto (h,k)$. Every element of HK can be written as hk where $h \in H$ and $k \in K$. The mapping is well defined; say $h_1k_1 = h_2k_2$ then we have $h_1h_2^{-1} = k_1k_2^{-1}$ but property 2 tells us that both sides are e. Thus we have $h_1 = h_2$ and $k_1 = k_2$.
Now say $\phi(h_1k_1h_2k_2) = \phi(h_1h_2k_1k_2) = (h_1h_2, k_1k_2) = (h_1, k_1)(k_1, k_2) = \phi(h_1, k_1)\phi(h_2, k_2)$

Well definedness gives us into and onto. And thus it is an isomorphism. □

Proposition 2.2 (Reformulation of recognition theorem)**.**
Let $H_1, H_2, \ldots, H_n$ be normal subgroups of G and assume that

1. $G = H_1H_2 \ldots H_n$
2. For each $i = 1, 2, \ldots n$ we have $H_i \cap (H_1 \ldots H_{i-1} \cap H_{i+1} \cap \ldots H_n) = \{e\}$

Then $G \cong H_1 \times H_2 \times \ldots \times H_n$

Theorem 2.3 (Primary Decomposition theorem)**.**
Let G be a finite abelian group of order n and the prime factorisation of n is $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ and let A_i be the Sylow-p_i subgroup of G for $i = 1, 2, \ldots n$. Then

$$G = A_1 \times A_2 \times \ldots A_k$$

Proof. Note that any subgroup of an abelian group is normal.

If $x \in G$ is an element whose order is a power of p_i then Sylow's theorem says that x is in Sylow-p_i subgroup. But there is only 1 Sylow-p_i subgroup since all are conjugates. $\Rightarrow x \in A_i$

Now, let $x \in G$ be an arbitrary element. Then $o(x)|n$ so let

$$o(x) = p_1^{\beta_1} p_2^{\beta_2} \ldots p_k^{\beta_k}$$

Then set

$$q_1 = \frac{o(x)}{p_1^{\beta_1}}, q_2 = \frac{o(x)}{p_2^{\beta_2}}, \cdots q_k = \frac{o(x)}{p_k^{\beta_k}}$$

Clearly q_is are coprime. So we have integers $a_1, a_2, \ldots a_k \in \mathbb{Z}$ such that

$$a_1q_1 + a_2q_2 + \ldots a_kq_k = 1$$

.

Then we have

$$\begin{aligned} x^1 &= x^{a_1q_1+a_2q_2+\dots a_kq_k} \\ x &= x^{a_1q_1} \cdot x^{a_2q_2} \cdots x^{a_kq_k} \end{aligned}$$

Moreover $(x^{a_iq_i})^{p_i^{\beta_i}} = e$ and thus $x^{a_iq_i} \in A_i$ for $i = 1, 2, \dots, k$

So we can say that $G = A_1A_2 \dots A_k$.

Now say $a_1 \in A_1$ and $a_1 = a_2a_3 \cdots a_k$ for $a_i \in A_i$. We also know that $ord(a_i) = p_i^{\alpha_i}$.

Then we have $a_1^{p_2^{\alpha_2} \dots p_k^{\alpha_k}} = e$ but p_is are all distinct therefore we have $a_1 = e$.

Now we can use the recognition theorem of direct products (proposition 2.2) to get the primary decomposition.

□

Now that we have proved that, we can focus on groups of prime power orders only.

Lemma 2.4.
Let G be a finite abelian p-group of order p^n. Then there exists powers $\beta_1, \beta_2, \dots, \beta_k$ with $\beta_1 \geq \beta_2 \geq \dots \geq \beta_k$ and $\beta_1 + \beta_2 + \dots + \beta_k = n$ such that

$$G \cong Z_{p^{\beta_1}} \times Z_{p^{\beta_2}} \times \cdots Z_{p^{\beta_k}}$$

Proof. Proof by induction. If $n = 1$ then $G =< e > \times < g >$ and we are done.

Let $g_1 \in G$ be an element of maximal order p^{β_1}, then $\beta_1 \leq m$. Let $Z_1 =< g_1 >$.

$$A/Z_1 =< g_2Z_1 > \times < g_3Z_1 > \times \cdots \times < g_kZ_1 >$$

where $| < g_iZ_1 > | = p^{\beta_i}$ and $\beta_2 \geq \beta_3 \geq \dots \geq \beta_k$. Since g_1 has maximal order $\therefore g_1 \geq g_2$

Fix i where $2 \leq i \leq r$ and consider $< g_iZ_1 > \leq A/Z_1$, $|g_iZ_1| = p^{\beta_i}$

We have $g_i^{p^{\beta_i}} \in Z_1 \Rightarrow g_i^{p^{\beta_i}} = g_1^{r_i}$ for some r_i

Then we have $(g_1^{r_i})^{p^{\beta_1-\beta_i}} = g_i^{p^{\beta_1}} = e$

This says that $p^{\beta_1} | r_ip^{\beta_1-\beta_i} \Rightarrow p^{\beta_i} | r_i \Rightarrow p^{\beta_i}d_i = r_i$

So $g_i^{p^{\beta_i}} = g_1^{r_i} = g_1^{\beta_id_i}$.

Now we set $x_i = g_i g_1^{-d_i}$ which gives us $x_i^{p^{\beta_i}} = g_i^{p^{\beta_i}} g_1^{-d_i p^{\beta_i}} = e$ and this is the smallest exponent that has this property.

So we can produce $x_2, x_3, \ldots, x_k \in G$ such that $ord(x_i) = p^{\beta_i}$. Set $Z_i =< x_i >$ for $2 \leq i \leq k$
To complete the proof it suffices to prove the following:

1. $G = Z_1 Z_2 \cdots Z_k$
2. For each $i = 1, 2, \ldots n$ we have $Z_i \cap (Z_1 \ldots Z_{i-1} \cap Z_{i+1} \cap \ldots Z_n) = \{e\}$

Let $g \in G$ then from the decomposition obtained above we can get powers $e_1, \ldots e_k$ such that

$$\begin{aligned} (g_2 Z_1)^{e_2} (g_3 Z_1)^{e_3} \cdots (g_k Z_1)^{e_k} &= g Z_1 \\ (g_2 Z_1)^{e_2} (g_3 Z_1)^{e_3} \cdots (g_k Z_1)^{e_k} &= g g' && \text{for some } g' \in Z_1 \end{aligned}$$

We also know that $g_i = x_i g_1^{d_i}$, thus we have

$$x_2^{e_2} \cdot x_3^{e_3} \cdots x_k^{e_k} = gh$$

for some $h \in Z_1$. We can thus choose e_1 to be such that $x_1^{e_1} = h^{-1}$. So finally we have

$$g = x_1^{e_1} x_2^{e_2} x_3^{e_3} \cdots x_k^{e_k}$$

and thus (1) is proven.

To prove (2) it suffices to prove that, if $x_1^{e_1} x_2^{e_2} \cdots x_k^{e_k} = e$ then $p^{\beta_i} | e_i$ for $i = 1, 2, \ldots, k$

Again note that $x_i = g_i g_1^{-d_i}$,

$$\begin{aligned} y(g_1^{e_1} g_2^{e_2} \cdots g_k^{e_k}) &= e && \text{for } y \in Z_1 \\ (g_1^{e_1} g_2^{e_2} \cdots g_k^{e_k}) Z_1 &= e Z_1 \end{aligned}$$

This gives us $p^{\beta_i} | e_i$ for $i = 2, 3, \ldots, k$ but this means $x_1^{e_1} = e \Rightarrow p^{\beta_1} | e_1$. Thus we are done by proposition 2.2 □

Lemma 2.5.
If G is a cyclic group of order p^α and $\phi : G \to G$ be a map such that $g \mapsto g^p$, then we claim that $|G| = p|G^p|$.

Proof. $G =< x >$ and $ord(x) = p^\alpha$

$G = \{1, x, x^2, \cdots, x^{p^\alpha - 1}\}$

$G^p = \{g^p | g \in G\}$. Clearly, $G^p \leq G$

$Ker(\phi) = \{g \in G \mid g^p = e\}$

If $g \in G \Rightarrow g = x^k$, and consider $x^{kp} = e \Rightarrow p^\alpha | kp$

This means $k = p^{\alpha-1}\lambda$ for $\lambda = 0, 1, 2, \ldots, p-1$

Thus we have obtained $Ker(\phi) = \{x^{(p^{\alpha-1})\lambda} \ : \ \lambda = 0, 1, \ldots, p-1\}$ and $|Ker(\phi)| = p$.

The first isomorphism theorem gives us $|G/Ker(\phi)| = |G^p| \Rightarrow |G| = p|G^p|$. □

Lemma 2.6 (Uniqueness)**.**
Suppose G is a finite abelian group of prime power orders. If $G = H_1 \times H_2 \times \cdots \times H_r = K_1 \times K_2 \times \cdots \times K_s$ where H_i, K_j are nontrivial cyclic subgroups with $|H_1| \leq \cdots \leq |H_r|$, $|K_1| \leq \cdots \leq |K_s|$ then $r = s$ and $|H_i| = |K_i|$.

Proof. Proceed by induction.

Suppose $|G| = p$, then it is fine.

Assume statement is true for finite abelian groups of order less than $|G|$.

Now,
$$G^p = H_1^p \times H_2^p \times \cdots \times H_{r'}^p$$
where r' is the largest integer i such that $|H_i| > p$

And,
$$G^p = K_1^p \times K_2^p \times \cdots \times K_{s'}^p$$
where s' is the largest integer i such that $|K_i| > p$

Now, $|G^p| < |G|$ and by induction hypothesis $r' = s'$,

And, $|H_i^p| = |K_i^p|$ for all $i = 1, 2, \ldots, r'$

But $|H_i| = p|H_i^p| \Rightarrow |H_i| = |K_i|$ for $i = 1, 2, \ldots, r'$ by lemma 2.5

All that remains to be proven is that the $\#H_i$ of order $p = \#K_i$ of order p.

$$|H_1||H_2|\cdots|H_{r'}|p^{r-r'} = |G| = |K_1||K_2|\cdots|K_{s'}|p^{s-s'}$$
And, $r' = s' \Rightarrow r = s$ and we are done. □

Thus we have proved the Fundamental Theorem of Finite Abelian Groups

3 Davenport's Constant and Some Results

This section will do the following:

1. State what the constant means.
2. Find results in rank-1.
3. Find results in rank-2 (specifically in C_p^2).
4. Use some inductive argument to get the result in rank-2 (in $C_m \oplus C_n$ with $m|n$).

> **Definition.**
> $\mathcal{D}(G)$ is the least positive integer such that all sequences over G of length $\mathcal{D}(G)$ has a zero sum subsequence.

Let us fix some notation before we move forward.

- The sequence S will be denoted in the multiplicative form $S = \prod_{i=1}^{l} g_i$.
- The sum of elements of S is denoted by $\sigma(S)$.
- A subsequence of S is denoted by $T \mid S$.

> **Definition.**
> We also introduce another constant $\mathcal{D}^*(G) = \sum_{i=1}^{r}(n_i - 1) + 1$ where n_is are obtained from $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$

3.1 Rank-1 results

A very trivial lower and upper bound on $\mathcal{D}(G)$ is as follows.

Theorem 3.1.
$\mathcal{D}^*(G) \leq \mathcal{D}(G) \leq |G|$

Proof. "first inequality"
$G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ and each $C_{n_i} =< e_i >$. Consider the sequence

$$S = e_1^{n_1-1} \cdot e_2^{n_2-1} \cdots e_r^{n_r-1}$$

Thus we have found a sequence of length $\mathcal{D}^*(G) - 1$ which is zero sum free therefore $\mathcal{D}(G)$ must be atleast 1 more than this, that is

$$\mathcal{D}(G) \geq \mathcal{D}^*(G) - 1 + 1 = \mathcal{D}^*(G)$$

"second inequality"

Consider the sequence $S = \prod_{i=1}^{l} g_i$ with $l \geq |G|$ and let $S_k = g_1 g_2 \ldots g_k$ where $k = 1, 2, \ldots, l$. If $\sigma(S_k) = 0$ then we are done otherwise we can always find $j < k$ such that $\sigma(S_j) = \sigma(S_k)$ (suppose all were distinct then there would be l distinct sums but only $|G|$ distinct elements in G). Consider $\sigma(g_{j+1} g_{j+2} \cdots g_k) = \sigma(S_k S_j^{-1}) = \sigma(S_k) - \sigma(S_j) = 0$ and thus we have found a subsequence whose terms sum to zero. Either way we find a $T \mid S$ such that $\sigma(T) = 0$. Hence, proved. □

Corollary 3.2.
If G is cyclic then $\mathcal{D}(G) = |G|$

3.2 Rank-2 results

Thus we have found Davenport constant for cyclic groups and now we move onto another elementary group which are p-groups.

Theorem 3.3.
Let $G = C_{p^{e_1}} \oplus C_{p^{e_2}} \oplus \cdots \oplus C_{p^{e_r}}$ be a finite abelian p-group. Then $\mathcal{D}^*(G) = \mathcal{D}(G)$

The proof of this theorem uses something called group ring which I will introduce. Also to note is that for the sake of this proof we will consider the group operation to be multiplication and not addition.

Definition.
Let G be a finite abelian group written multiplicatively and R be a commutative ring with 1. Now consider the formal sum where only finitely many terms are non-zero.

$$R[G] = \left\{ \sum_{g \in G} a_g g : \ a_g \in R \right\}$$

If $x, y \in R[G]$ such that $x = y$ iff $a_g = b_g \ \forall \ g \in G$.
Note that $G \subset R[G]$, $1 \in R[G]$. If we define addition and multiplication then we are done.

$$x + y = \sum_{g \in G} (a_g + b_g) g$$

$$x \cdot y = \sum_{g \in G} (c_g g) \text{ where } c_g = \sum_{g_1, g_2 \in G, g_1 g_2 = g} a_{g_1} b_{g_2}$$

Definition.
$N^k(S) = \#\{T \mid S \ : \ |T| = k, \text{zero sum subsequence}\}$

Lemma 3.4.
If $|S| \geq \mathcal{D}^*(G)$ then

$$1 - N^1(S) + N^2(S) - \cdots (-1)^k N^k(S) \equiv 0 \pmod{p}$$

Lemma 3.5 (*).
[Ols69a] Let the sequence $S = \prod_{i=1}^{l} g_i$ be over G such that $l \geq \mathcal{D}^*(G)$. Then

$$(1 - g_1)(1 - g_2) \cdots (1 - g_l) \in p\mathbb{Z}[G]$$

Proof. Suppose $g \in G$ and $g = uv$ then we can write $1 - g = 1 - uv = (1 - u) + u(1 - v)$.
$G = C_{p^{e_1}} \oplus C_{p^{e_2}} \oplus \cdots \oplus C_{p^{e_r}}$ and let $C_{p^{e_i}} = < x_i > \ \forall i = 1, 2, \cdots l$.
For any $g \in G$ we have $g = x_1^{a_1} \cdot x_2^{a_2} \cdots x_r^{a_r}$.

$$\begin{aligned} 1 - g &= 1 - x_1^{a_1} \cdot x_2^{a_2} \cdots x_r^{a_r} \\ &= (1 - x_1) + x_1(1 - x_1) + x_1^2(1 - x_1) \cdots x_1^{a_1 - 1}(1 - x_1) \\ &+ x_1^{a_1}(1 - x_2) + x_1^{a_1} x_2(1 - x_2) + x_1^{a_1} x_2^2(1 - x_2) \cdots x_1^{a_1} x_2^{a_2 - 1}(1 - x_2) \\ &\cdots \\ &\cdots \\ &+ x_1^{a_1} x_2^{a_2} \cdots x_{r-1}^{a_{r-1}}(1 - x_r) + x_1^{a_1} x_2^{a_2} \cdots x_{r-1}^{a_{r-1}} x_r(1 - x_r) + \cdots + x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r - 1}(1 - x_r) \\ &\in \mathbb{Z}[G] \end{aligned}$$

Now if we look at $(1 - g_1)(1 - g_2) \cdots (1 - g_l) = \sum g J_g$ where $J_g = (1 - x_1)^{c_1}(1 - x_2)^{c_2} \cdots (1 - x_r)^{c_r}$. A crucial observation is that $c_1 + c_2 + \cdots + c_r \geq l \geq \sum_{i=1}^{r} (p^{e_i} - 1)$.
This tells us that there must be a j such that $c_j \geq p^{e_j}$.
Consider $(1 - x_j)^{c_j} = (1 - x_j)^{p^{e_j}}(1 - x_j)^{c_j - p^{e_j}} = (1 - p \cdot \text{ terms } (-1)^{p^{e_j}})$(group ring element). If $p = 2$ then it becomes 2(group ring element) and if p is odd then also it becomes p(group ring element). Therefore $(1 - x_j)^{c_j} \in p\mathbb{Z}[G]$ or equivalently

$$(1 - g_1)(1 - g_2) \cdots (1 - g_l) \equiv 0 \pmod{p}$$

□

Proof of lemma 3.4. Let the sequence be $S = \prod_{i=1}^{l} g_i$ with $l \geq \mathcal{D}^*(G)$.
Note that $(1 - g_1)(1 - g_2) \cdots (1 - g_l) = \sum a_g g$. The lemma 3.5 tells us that $a_g \equiv 0 \pmod{p} \forall \, g \in G$.

Now consider $g = 1$ and therefore look at a_1. First let us rewrite the LHS

$$1 - \sum_{i=1}^{l} g_i + \sum_{i<j} g_i g_j - \cdots (-1)^l g_1 \cdots g_l$$

Combinatorially what it means is that we are trying to find elements in the sum such that product of g_is equal 1. The contribution for a_1 can come from any of the terms in the above sum and thus we are trying to find zero sum sequences in the above sum and thus we have established the required relation. ☐

Proof of theorem 3.3. Suppose that the sequence has length more than $\mathcal{D}^*(G)$ but it is zero-free then we have $1 \equiv 0 \pmod p$ and therefore we get $\mathcal{D}(G) \leq \mathcal{D}^*(G)$ but we also know that $\mathcal{D}(G) \geq \mathcal{D}^*(G)$ from theorem 3.1. Thus we have $\mathcal{D}(G) = \mathcal{D}^*(G)$. ☐

Corollary 3.6.
$\mathcal{D}(C_p^r) = r(p-1)+1$

Conjecture 3.1.
$\mathcal{D}(C_n^r) = r(n-1)+1$

Theorem 3.7 (Girard)**.**
[Gir18] $\mathcal{D}(C_n^r)$ is asymptotically bounded by nr

Definition.
$\eta(G)$ is the least positive integer l such that any given sequence S of length $|S| \geq l$ over G satisfying $N^k(S) \geq 1$ for some $1 \leq k \leq exp(G)$

Lemma 3.8.
$\eta(G) = 3p-2$ that is any sequence of length $\geq 3p-2$ over G has a zero sum subsequence of length at most p

Proof. Let $S = \prod_{i=1}^{l} g_i$ be over C_p^2. We observe C_p^2 as a subgroup of C_p^3 such that the third term is 0. Consider the new sequence $T = \prod_{i=1}^{l}(g_i, 1)$. We know that $\mathcal{D}(C_p^3) = 3p-2$ from theorem 3.3 therefore we have a zero sum subsequence $T' \mid T$ such that $\sigma(T') = 0$. This means $|T'| \equiv 0 \pmod p$ so $|T'| = p, 2p$. If $|T'| = p$ we are done. If $|T'| = 2p$, we know that $2p-1 = \mathcal{D}(C_p^2)$ we will find a zero sum subsequence T'' of length less than $2p$. So either this subsequence or its complement in T' will have length less than p and we are done. ☐

Theorem 3.9 (Olson)**.**
Let $G = C_m \times C_n$ with $m \mid n$. Then $\mathcal{D}(G) = m + n - 1 = \mathcal{D}^*(G)$

Proof. Let p be the prime that divides both m, n then let $m = pm_1$ and $n = pn_1$. If $n_1 = 1$ then we have $\mathcal{D}(C_p^2) = 2p - 1$. So we will assume that we have $n_1 \geq 2$.

Proof by induction.

Consider $H \leq C_m$ and $K \leq C_n$ with $[H : C_m] = p = [K : C_n]$

Now take $Q = H \times K$ with $|Q| = m_1 n_1$.

Let $S = \prod_{i=1}^{l} g_i$ be the sequence over G with $l \geq m + n - 1$

Take the quotient G/Q and consider the sequence $T = \prod_{i=1}^{l} (g_i + Q)$. Note that

1. $G/Q \cong C_p^2$ and $\mathcal{D}(C_p^2) = 2p - 1$ therefore we can find a zero sum sequence, call it T_1 with length at most p

2. $m + n - 1 = p(m_1 + n_1 - 2) + 2p - 1$

Now throw this away and look at the rest, if the condition (2) is satisfied then we can again throw away. Say we throw away $T_1, T_2, \cdots T_u \in Q$. Then we have thrown away at most $m_1 + n_1 - 2$ sequences and thus we are left with atleast $2p - 1$ elements in $T \backslash \{T_1, \cdots, T_u\}$ but $\mathcal{D}^*(C_p^2) = 2p - 1$ therefore we can find another zero sum subsequence say $T' \in Q$. Now look at the sequence $S' = \sigma(T')\sigma(T_1) \cdots \sigma(T_u)$ of length $m_1 + n_1 - 1$. Therefore by induction hypothesis we find a subsequence in S' call it S'' such that $\sigma(S'') = 0$. So we have found zero sum subsequence in S itself. And we are done. □

4 Erdös-Ginzberg-Ziv Constant

This section will do the following:

1. State what the constant means.

2. Find results in rank-1.

3. Find results in rank-2 (specifically in C_p^2).

4. Use some inductive argument to get the result in rank-2 (in $C_m \oplus C_n$ with $m|n$).

Recall what the constant is

> **Definition.**
> $s(G$ is the smallest integer such that any sequence S over G such that $|S| \geq s(G)$ has a

zero sum subsequence of length equals $\exp(G)$

4.1 Erdös- Ginzberg-Theorem (Rank-1)

Theorem 4.1 (Erdös-Ginzberg-Ziv Theorem)**.**
A sequence of $2n-1$ elements over C_n has a zero sum subsequence of length n.

First consider the sequence $0^{n-1}e^{n-1}$ of length $2n-2$ where $C_n =< e >$. It is clearly n-subsequence zerosum free and thus a trivial lower bound is $s(G) \geq 2n-1$.

Note 4.2.
We will see that it is enough to prove the statement for $n = p$ where p is a prime.

Why is that ? Well, say it were true for a prime p and $n = pm$. The proof will be through induction. Consider the sequence

$$S = g_1 g_2 \cdots g_{2n-1}$$

Since the result is true for prime p, for every sequence of length $2p-1$ we have a subsequence whose sum is divisible by p. Remove this zero sum p-subsequence from S, call it I_1 and continue removing in this manner. We find that we can remove $I_1, I_2, \ldots I_{2m-1}$, since after we remove $2m-2$ p-subsequence we are left with atleast $2p-1$ elements which again has a subsequence of length p divisible by p which can be removed. Now consider the sequence

$$T = \prod_{i=1}^{2m-1} \frac{\left(\sum_{j \in I_i} g_j\right)}{p}$$

Now we see that T has $2m-1$ elements and therefore by the induction hypothesis we know that there exists a subsequence of length m such that it is divisible by m. The corresponding I_js each of length p should then give us the relevant n-subset which is 0 modulo n.

So, it is enough to prove the following proposition:

Proposition 4.3.
For a prime p and a sequence of elements $g_1, g_2, \ldots, g_{2p-1}$ over C_p, there is an $I \subset \{1, 2, \ldots 2p-1\}$ such that $|I| = p$ and $\sum_{i \in I} g_i \equiv 0 \pmod{p}$

There are many proofs of this proposition and they can be found in [AD93]. I will present two of them in this report.
Proof 1: Requires the following lemma

Lemma 4.4 (Cauchy-Davenport)**.**
Let A and B be two non-empty subsets of C_p then $|A+B| \geq \min\{p, |A|+|B|-1\}$

Proof. If □

Now let's prove the proposition.
If $\exists\ g_i$ which appears atleast p times in the sequence S, then we can clearly find the zero sum subsequence of length p. So let us assume that there is no g_i which appears more than $p-1$ times. Rewrite the g_i s in the following manner:

$$g_1\ g_2\ \cdots g_{p-1}\ g_{2p-1}$$

$$h_1\ h_2\ \cdots h_{p-1}$$

where $g_i \neq h_i$ for all $i = 1, 2, \ldots, p-1$.
Applying Cauchy-Davenport lemma we have $|\{g_1, h_1\} + \{g_2, h_2\}| \geq 3$. Using the lemma repeatedly we get $\left|\sum_{i=1}^{s}\{g_i, h_i\}\right| \geq 2s - (s-1) = s+1$ for $s = 1, 2, \ldots, p-1$
In particular we have $\left|\sum_{i=1}^{p-1}\{g_i, h_i\}\right| \geq p$
Equivalently we can say that $\sum_{i=1}^{p-1}\{g_i, h_i\} = C_p$
This means that $-g_{2p-1} \in \sum_{i=1}^{p-1}\{g_i, h_i\}$, that is $\exists\ f_i \in \{g_i, h_i\} \forall i = 1, 2, \ldots, p-1$ such that
$f_1 + f_2 + \cdots + f_{p-1} = -g_{2p-1}$ and thus we have found the zero sum subsequence of length p. □

Proof 2: Proof using Davenport's constant.
Recall that for p-groups $\mathcal{D}(G) = \mathcal{D}^*(G)$.
Consider the sequence $S = g_1, g_2, \ldots, g_{2p-1}$ over C_p.
Take the sequence $T = (g_1, 1), (g_2, 1), \ldots, (g_{2p-1}, 1)$ over $C_p \oplus C_p$. We know that $\mathcal{D}(C_p^2) = 2p-1$ and therefore we have a subsequence $T'|T$ such that $\sigma(T') = 0$ but this means $\underbrace{1+1+\cdots+1}_{k \text{ times}} \equiv 0 \pmod p$ which is possible only when k is multiple of p and therefore $k = p$ and we are done. □

So we have proved $s(C_n) = 1$. So we are done with rank 1 groups. Now we remember how we moved to rank 2 for Davenport's constant. We found results for C_p^2 and then used some inductive argument to go to $C_m \oplus C_n$. This is exactly what we try to do here as well.

We consider the higher order analogue of EGZ theorem, that is we try to state a similar result in $C_n \oplus C_n$.

4.2 Rank-2 results

Theorem 4.5 (Kemnitz conjecture)**.**
Given a sequence S of $4n-3$ lattice points over $G \cong C_n \oplus C_n$ we claim that there is a subsequence of length n whose elements sum to zero.
Equivalently $s(G) = 4n-3$

Note that the sequence $(0,0)^{n-1}(1,0)^{n-1}(0,1)^{n-1}(1,1)^{n-1}$ of length $4n-4$ does not have a zero sum subsequence of length n and therefore $s(G) \geq 4n-3$.

Proposition 4.6.
It suffices to prove the theorem for $n = p$ where p is prime.

Proof. Why is that ? Well, say it were true for a prime p and $n = pm$. The proof will be through induction. Consider the sequence

$$S = g_1 g_2 \cdots g_{4n-3}$$

Since the result is true for prime p, for every sequence of length $4p-3$ we have a subsequence whose sum is divisible by p. Remove this zero sum p-subsequence from S, call it I_1 and continue removing in this manner. We find that we can remove $I_1, I_2, \ldots I_{4m-3}$, since after we remove $4m-4$ p-subsequence we are left with atleast $4p-3$ elements which again has a subsequence of length p divisible by p which can be removed. Now consider the sequence

$$T = \prod_{i=1}^{4m-3} \frac{\left(\sum_{j \in I_i} g_j\right)}{p}$$

Now we see that T has $4m-3$ elements and therefore by the induction hypothesis we know that there exists a subsequence of length m such that it is divisible by m. The corresponding I_js each of length p should then give us the relevant n-subset which is 0 modulo n. $\square$

Therefore effectively we just have to prove that

Proposition 4.7.
$s(C_p^2) = 4p-3$

The lower bound is known. The first upper bound was due to [AD95] who proved that $s(G) \leq 6p-5$ for all primes p and $s(G) \leq 5p-2$ for large p. An improvement was provided by [Rón00] who proved that $s(G) \leq 4p-2$. So we were left with two possibilities $s(G) = 4p-3$ or $4p-2$.
It was finally proved independently by Reiher [Rei07] and di Fiore. Further developments on this is found in [GHZ16]

4.2.1 Proof by Reiher

Here I will lay out the proof given by Reiher in details.

Note 4.8 (KEY IDEA).
Let $G = C_p^2$ and consider $G \oplus C_p$. We see that $\mathcal{D}^*(G \oplus C_p) = 3p - 2$. If $C_p =< e >$ where $ord(e) = n$ then we can write $G \oplus C_p = G\oplus < e >$.

For a sequence $S = \prod_{i=1}^{l} g_i$ over G define $\phi(S)$ be the sequence over $G\oplus < e >$ where $\phi :$ $G \rightarrow G\oplus < e >$ such that $\phi(g) = (g, e)$. That is $\phi(S = \prod_{i=1}^{l} g_i\ ) = (g_1, e)(g_2, e)\cdots(g_l, e)$. Thus any zero sum in $\phi(S)$ will have length $\equiv 0 \pmod p$

Lemma 4.9.
If $|S| = 3p - 3$, then

$$1 - N^{p-1}(S) - N^p(S) + N^{2p-1}(S) + N^{2p}(S) \equiv 0 \pmod p$$

Proof. Consider the sequence $0S$ of length $3p - 2$ over G and
Consider the sequence $\phi(0S)$ of length $3p - 2$ over $G \oplus C_p$ and

Then the zero sums of even length $= N^0(\phi(0S)) + N^{2p}(\phi(0S)) = 1 + N^{2p-1}(S) + N^{2p}(S)$
Then the zero sums of odd length $= N^p(\phi(0S)) = N^{p-1}(S) + N^p(S)$

Now from lemma 3.4 we have the claim. ☐

Lemma 4.10.
If $|S| = 3p - 2$ or $|S| = 3p - 1$, then

$$1 - N^p(S) + N^{2p}(S) \equiv 0 \pmod p$$

Proof. From lemma 3.4 it is trivially true. ☐

Lemma 4.11.
If $|S| = 3p - 2$ or $|S| = 3p - 1$, then

$$N^p(S) \equiv 0 \pmod p \Rightarrow N^{2p}(S) \equiv -1 \pmod p$$

Proof. A direct consequence of lemma 4.10. ☐

Lemma 4.12 (Alon, Dubiner).
If S contains exactly $3p$ elements whose sum is $\equiv 0 \pmod p$ then $N^p(S) > 0$

Proof. Suppose $N^p(S) = 0$ then it is easy to see that $N^p(S/g) = 0$ for some arbitrary element $g \mid S$. But we also observe that $|S/g| = 3p-1$ and from lemma 4.11 we have $N^{2p}(S/g) \equiv -1 \pmod p \Rightarrow N^{2p}(S/g) > 0 \Rightarrow N^{2p}(S) > 0$. We also note that $\sigma(S) \equiv 0 \pmod p$ which gives us $N^{2p}(S) \equiv N^p(S) > 0$ and thus $N^p(S) > 0$. □

Lemma 4.13.
If $|S| = 4p-3$, then

1. $-1 + N^p(S) - N^{2p}(S) + N^{3p}(S) \equiv 0 \pmod p$
2. $N^{p-1}(S) - N^{2p-1}(S) + N^{3p-1}(S) \equiv 0 \pmod p$
3. $3 - 2N^{p-1}(S) - 2N^p(S) + N^{2p-1}(S) + N^{2p}(S) \equiv 0 \pmod p$

Proof.

1. Consider the sequence S of length $4p-3$ over G and
Consider the sequence $\phi(S)$ of length $4p-3$ over $G \oplus C_p$ and

   Then the zero sums of even length $= N^0(\phi(S)) + N^{2p}(\phi(S))$
   Then the zero sums of odd length $= N^p(\phi(S)) + N^{3p}(\phi(S))$

   Now from lemma 3.4 we have the claim.

2. Consider the sequence $0S$ of length $4p-2$ over G and
Consider the sequence $\phi(0S)$ of length $4p-2$ over $G \oplus C_p$ and

   Then the zero sums of even length $= N^0(\phi(0S)) + N^{2p}(\phi(0S)) = 1 + N^{2p-1}(S) + N^{2p}(S)$
   Then the zero sums of odd length $= N^p(\phi(0S)) + N^{3p}(\phi(0S)) = N^{p-1}(S) + N^p(S) + N^{3p-1}(S) + N^{3p}(S)$

   Now from lemma 3.4 and item 1 we have the claim.

3. Let us fix some notation before we move ahead.
If $S = \prod_{i=1}^{4p-3} g_i$ and $J = \{1, 2, \dots, 4p-3\}$, for $I \subset J$ we have $S_I = \prod_{i \in I} g_i$.

   Consider the congruence in lemma 4.9

$$1 - N^{p-1}(S) - N^p(S) + N^{2p-1}(S) + N^{2p}(S) \equiv 0 \pmod p$$

Now,

$$\Rightarrow \sum_{I\subset J,|I|=3p-3} (1 - N^{p-1}(S_I) - N^p(S_I) + N^{2p-1}(S_I) + N^{2p}(S_I)) \equiv 0 \pmod p$$

$$\Rightarrow \binom{4p-3}{3p-3} - \binom{3p-2}{2p-2} N^{p-1}(S) - \binom{3p-3}{2p-3} N^p(S) + \binom{2p-2}{p-2} N^{2p-1}(S) + \binom{2p-3}{p-3} N^{2p}(S) \equiv 0 \pmod p$$

$$\Rightarrow \binom{4p-3}{p} - \binom{3p-2}{p} N^p(S) - \binom{3p-3}{p} N^p(S) + \binom{2p-2}{p} N^{2p-1}(S) + \binom{2p-3}{p} N^{2p}(S) \equiv 0 \pmod p$$

If $a \in \mathbb{N}$ and $b \in \{1, 2, \dots p\}$ we have the following

$$\begin{aligned}(p-1)!\binom{ap-b}{p} &\equiv \frac{1}{p}\prod_{i=0}^{p-1}(ap-b-i) \pmod p\\ &\equiv (a-1)\prod_{i=0,i\neq p-b}^{p-1}(ap-b-i) \pmod p\\ &= (a-1)(p-1)!\end{aligned}$$

$\therefore \binom{ap-b}{p} \equiv (a-1) \pmod p$

So we have $3 - 2N^{p-1}(S) - 2N^p(S) + N^{2p-1}(S) + N^{2p}(S) \equiv 0 \pmod p$

□

Lemma 4.14.
If $|S| = 4p-3$ and $N^p(S) \equiv 0 \pmod p$, then

$$N^{p-1}(S) \equiv N^{3p-1}(S) \pmod p$$

Proof. Let χ be the number of partitions of $S = I_1 \cup I_2 \cup I_3$ satisfying

$$|I_1| = p-1, |I_2| = p-2, |I_3| = 2p$$

and

$$\sigma(I_1) = 0 = \sigma(I_3), \sigma(I_2) = \sigma(S)$$

Run through all permissible I_1 and count permissible I_2 in S/I_1 to determine χ

$$\chi \equiv \sum_{I_1} N^{2p}(S/I_1) \equiv \sum_{I_1} -1 \equiv -N^{p-1}(S) \pmod p$$

from lemma 4.11.

Counting I_2 another way :

$$\chi \equiv \sum_{I_1\cup I_3} N^{2p}(S/I_1) \equiv \sum_{I_1\cup I_3} -1 \equiv -N^{3p-1}(S) \pmod p$$

from lemma 4.11. And we are done.

∎

Proof of Kemnitz conjecture. If $p = 2$ then $|S| = 5$ and thus we would find $g^2|S$ and we would be done. Therefore assume p is odd.

Let us add up the three congruences in item 1,item 2,item 3 along with the assumption in lemma 4.14 to get $2 - N^p(S) + N^{3p}(S) \equiv 0 \pmod p$ OR $N^{3p}(S) \equiv 2 \pmod p$ but lemma 4.12 tells us that $N^p(S) \neq 0$ and we are done. ∎

4.2.2 Structural insights that contain the proof due to Reiher

Lemma 4.15.
Let $G = H \oplus C_n$ with $\exp(H)|n$ and $\mathcal{D}(H) \leq n$, then

$$s(G) \geq 2(\mathcal{D}(H) - 1) + 2(n-1) + 1$$

Proof. If we can find a zero sum free sequence of length $2(\mathcal{D}(H) - 1) + 2(n-1)$, we are done. Let T be a zero free sequence over H of length $\mathcal{D}(H) - 1$.
Write $G = H \oplus C_n = H \oplus < e >$ with $ord(e) = n$
Then consider the sequence $S = T(T+e)e^{n-1}0^{n-1}$, $T = 2(\mathcal{D}(H) - 1) + 2(n-1)$. I claim that this is n-subsequence zero free. Why?

- We cannot get a n-subsequence zero sum if we take $e^{n-1}, 0^{n-1}, T$ by themselves.
- $T, T+e$ are zero free because $\mathcal{D}(H) \leq n$; T, e is also zero free because of same reason; $T, 0; e, 0; T+e, 0$ are zero free trivially; $T+e, e$ is also zero free because we will always end up taking a subsequence of T which is n-subsequence zero free.
- $T, T+e, e$ is zero free we will go beyond the requirement of finding a n-subsequence zero sum

Hence, we are done. ∎

Remark 4.16.
1. $\eta(G) \geq 2(\mathcal{D}(H) - 1) + n$
   Let T be a zero free sequence over H of length $D(H) - 1$.
   Consider the sequence $S = T(T+e)e^{n-1}$, $|S| = 2(\mathcal{D}(H) - 1) + (n-1)$
   This does not have a zero sum of length less than $\exp(G) = n$ and so we are done.
2. One also has $s(G) \geq \eta(G) + \exp(G) - 1$

Conjecture 4.1 (Gao)**.**
In the remark 2 above, equality always holds, that is $s(G) = \eta(G) + \exp(G) - 1$

Now that we have lower bound we try to find an upper bound. A relatively general upper bound is presented by Schmid [SZ10]. He uses motivation from the structural insights provided by Savchev-Chen [SC05].

Lemma 4.17.
If G is a finite abelian p-group with $\exp(G) = n$ and $\mathcal{D}(G) \leq 2n-1$, then

$$2\mathcal{D}(G) - 1 \leq \eta(G) + n - 1 \leq s(G) \leq \mathcal{D}(G) + 2n - 2$$

In particular, if $\mathcal{D}(G) = 2n-1$ we have $s(G) = \eta(G) + n - 1 = 4n - 3$

Thus we have obtained Kemnitz conjecture as a special case of our lemma.
Now that we have $s(C_n), s(C_p^2)$ we move to general rank 2 group and prove the following result:

5 More results on Davenport constant and other constants

This section will contain the following:

1. $\mathcal{D}(G)$ where $G = H \oplus C_{p^k}$ is a $p-group$
2. $\mathcal{D}(G)$ for $G \cong C_2 \oplus C_2 \oplus C_{2n}$
3. $\mathcal{D}(G)$ for $G \cong C_2 \oplus C_3 \oplus C_{3d}$
4. Values of $\eta(G), s(G)$ other than the ones obtained so far.

Till now we have calculated Davenport's constant for cyclic groups C_n, p-groups, rank-2 groups. Now we move up a rank and try to find the constant for certain special type of rank-3 groups.

5.1 Davenport's constant ($\mathcal{D}(G)$)

So we have seen earlier that $\mathcal{D}(G) = \mathcal{D}^*(G)$ for rank-2. The groups that have this property are known as *rank-2 like groups*.

Proposition 5.1.
Let $G = H \oplus C_{p^k}$ be a p-group with $\mathcal{D}(H) = p^k$ then

$$\mathcal{D}(H \oplus C_{np^k}) = \mathcal{D}^*(H \oplus C_{np^k}) = \mathcal{D}^*(H) + (np^k - 1) = \mathcal{D}(H) + np^k - 1 = p^k(n-1) - 1$$

where $n \in \mathbb{N}$

Proof. We can show that $\eta(G) \geq 3p^k - 2$ from item 1

Case ($n = 1$) It is trivially true.
Case ($n \geq 2$) Note that $np^k + p^k - 1 = (n-2)p^k + 3p^k - 1$
Let $Q \cong C_n$ be a subgroup of G such that $G/Q \cong H \oplus C_{p^k}$.
Consider sequence S over G of length $\mathcal{D}^*(G)$ and its projection $\phi(S)$ in G/Q. Then it has atleast n zero sum subsequences in Q so we will get zero sum subsequence in the original sequence S as well. □

5.1.1 Results for $C_2 \oplus C_2 \oplus C_{2n}$

Lemma 5.2.
A sequence S over C_n of length $n-1$ is short zero sum free if and only if it is of the form $S = g^{n-1}$ for some $g \in C_n$ and $ord(g) = n$.

Proof. '$\Leftarrow$' If $S = g^{n-1}$ for some $g \in G$ with $ord(g) = n$, then any subsequence of S will fail to give a zero sum and thus there are no short zero sum sub-sequences of S.
'$\Rightarrow$' Suppose $S = g_1 g_2 \cdots g_{n-1}$ is short zero sum free with $g_1 \neq g_2$
Consider $S_k = g_1 + g_2 + \cdots + g_k$ where $k = 1, 2, \ldots, n-1$.
Let $U = \{S_k : k = 1, 2, \ldots, n-1\}$

1. $0 \notin U$
2. $|U| \leq n-1$
3. $|U| \geq n-1$ becuase of the zero-free condition.

Combining 2, 3 gives us $|U| = n-1$ but $g_2 \in U$ which is a contradiction, therefore all elements are equal. □

Lemma 5.3.
For $G = C_2^r$ the following is true :

$$\eta(G) = 2^r, s(G) = 2^r + 1$$

Proof. First consider $\eta(C_2^r)$. Consider a sequence over C_2^r of length $2^r - 1$ which has a zero sum subsequence of length at most 2, the only possibilities are either it contains the 0-element or one of the element has to repeat. Suppose the sequence does not have the 0-element and no repeating elements, then take all the possible nonzero elements in the sequence (we have $2^r - 1$ of them). To get a zero sum subsequence of length 2 we will have to repeat one of the nonzero element, and therefore $\eta(G) = 2^r$
Similar reasoning but the additional restriction that the zero sum subsequence needs to have a length 2 means that we have to repeat one element and therefore $s(G) = 2^r + 1$ □

Note 5.4.
If $n = even = 2m$ then we are in "rank 2- like groups", $C_2 \oplus C_2 \oplus C_{4m}$

Theorem 5.5 (Baayen, van Lint)**.**
For odd n, we have $\mathcal{D}(C_2 \oplus C_2 \oplus C_{2n}) = \mathcal{D}^*(C_2 \oplus C_2 \oplus C_{2n}) = 2n + 2$

Proof. We will show that we can find a zero-sum sequence if a sequence of length $2n+2$ is taken.

Let $K \leq C_2 \oplus C_2 \oplus C_{2n}$ such that $K \cong C_n$. Let S be the sequence of length $2n+2$ over $C_2 \oplus C_2 \oplus C_{2n}$

Consider the quotient $C_2 \oplus C_2 \oplus C_{2n}/K \cong C_2^3$ and sequence $T = S + K$ which means if $S = g_1 \cdots g_{2n+2}$ then $T = (g_1 + K) \cdots (g_{2n+2} + K)$

Note that $2n+2 = 2(n-3) + \eta(C_2^3)$ using lemma 5.3

Thus we can extract $n-2$ disjoint subsequences, call them $S_1, S_2, \cdots, S_{n-2}$ with $|S_i| \leq 2$ and $\sigma(S_i) \in K$ for $i = 1, 2, \ldots, n-2$

The remaining sequence $S' = S(S_1 S_2 \cdots S_{n-2})^{-1}$ has length atleast 6.

If it contains a zero element or an element repeated more than once then also we can find 2 zero sum subsequences.

If the length is atleast 7 with no zero element and no repeating element then we can find two more subsequences whose sum lies in K because of the nature of elementary 2-group C_2^3, we can always find 3 elements that have zero sum and the remaining elements are atleast 4 and $\mathcal{D}(C_2^3) = 4$ and thus we can find another zero sum subsequence.

Assume that $|S'| = 6$ and all elements in projection of S' to C_2^3 are distinct and nonzero. Then we can again exploit the fact that it is an elementary 2 group and choose basis e_1, e_2, e_3 ($C_2 \oplus C_2 \oplus C_{2n}/K \cong < e_1 > \oplus < e_2 > \oplus < e_3 >$) such that the 6 elements can be written as

$$e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3$$

The corresponding elements in S' are say $g_1, g_2, g_3, g_{12}, g_{23}, g_{13}$

Note that we cannot have disjoint zero sum sequences anymore but we can have multiple zero sum sequences with the above basis.

Let $U | S'$ be such that $\sigma(U) \in K$.

Consider the sequence in $K \cong C_n$, $\sigma(S_1)\sigma(S_2)\cdots\sigma(S_{n-2})\sigma(U)$, length is $n-1$. If this has a zero sum subsequence we are done. Assume to the contrary that it is zero sum free and from our lemma it is possible only when all the elements are equal due to lemma 5.2, i.e., there is a $g_0 \in K$ such that $\sigma(U) = g_0$ for each $U | S'$ with $\sigma(U) \in K$.

Let us see what are the possible Us

g_1	g_2		g_{12}			U_1
g_1		g_3		g_{13}		U_2
	g_2	g_3			g_{23}	U_3
			g_{12}	g_{13}	g_{23}	U_4
g_1		g_3	g_{12}		g_{23}	V_1
g_1	g_2			g_{13}	g_{23}	V_2
	g_2	g_3	g_{12}	g_{13}		V_3

We find that $\sigma(U_i) = g_0 \in K$ and $\sigma(V_j) = g_0 \in K$ and also $|U_i| = 3, |V_j| = 4$

We also find that $4g_0 = U_1U_2U_3U_4 = V_1V_2V_3 = 3g_0 \Rightarrow g_0 = 0$ which is a contradiction and thus the sequence has a zero sum subsequence in K and thus a zero sum subsequence exists in the original sequence. $\square$

Theorem 5.6.
For $m|n$ one has $\mathcal{D}(C_2 \oplus C_{2m} \oplus C_{2n}) = \mathcal{D}^*(C_2 \oplus C_{2m} \oplus C_{2n})$

Proof. Classical result under some technical result on zero sum sequences over rank-2 groups. $\square$

Question 5.7.
What happens to $C_2^{r-1} \oplus C_{2n}$

Proposition 5.8.
$\mathcal{D}(C_2^4 \oplus C_{2n}) > \mathcal{D}^*(C_2^4 \oplus C_{2n}) = 2n + 4$ for odd n

Proof. I will produce a zero-sum free sequence of length $2n + 4$

Let $C_2^4 \oplus C_{2n} \cong < e_1 > \oplus < e_2 > \oplus < e_3 > \oplus < e_4 > \oplus < g >$ where $ord(e_i) = 2$ and $ord(g) = 2n$

IDEA: Take a $K \cong C_n$ a subgroup and consider the quotient. Then extract sequences whose sum lies in K. We wish to construct a sequence which just falls short of being zero in C_n. So we control the gs. We want to have one spot empty in the cyclic group (for our control), we want atmost $n - 2$ sequences in the cyclic group so we can have atmost $2(n - 2) + 1$ times g. Take the sequence to be g^{2n-3}. We want to find 7 more elements so that we can have zero-sum free sequence of desired length. Now we construct the remaining 7 elements.

First focus just on C_2^4. We wish to construct 7 elements that contains zero sum subsequences.

e_1				S_1
	e_2			S_2
		e_3		S_3
			e_4	S_4
e_1	e_2	e_3		S_5
e_1	e_2		e_4	S_6
	e_2	e_3	e_4	S_7

Note that each zero-sum subsequence has length 4. Now we want to add gs to the sequences such that $4 \times (xg) \equiv 2g \pmod{2n}$, so $x = (n+1)/2$ works. Therefore our seven elements are as follows:

e_1				$\frac{n+1}{2}g$	S_1
	e_2			$\frac{n+1}{2}g$	S_2
		e_3		$\frac{n+1}{2}g$	S_3
			e_4	$\frac{n+1}{2}g$	S_4
e_1	e_2	e_3		$\frac{n+1}{2}g$	S_5
e_1	e_2		e_4	$\frac{n+1}{2}g$	S_6
	e_2	e_3	e_4	$\frac{n+1}{2}g$	S_7

Thus we have constructed a sequence $S = g^{2n-3}S_1S_2\cdots S_7$of length $2n+4$ which is zero free, and we are done. $\square$

5.1.2 Results for $C_3 \oplus C_3 \oplus C_{3d}$

[Gau]

Question 5.9 (Open problems)**.**
These observations tempt us to ask the following questions:

1. Does $\mathcal{D}(C_n^r) = \mathcal{D}^*(C_n^r)$ always hold?

2. Does $\mathcal{D}(G) = \mathcal{D}^*(G)$ always hold?

Further results in rank-3 can be found in [Zak19a]

5.2 $\eta(G)$ and $s(G)$

5.2.1 Certain results on $\eta(G)$ and $s(G)$

Theorem 5.10.
If $G = C_{n_1} \oplus C_{n_2}$ is a finite abelian group with $n_1|n_2$ then

$$s(G) = 2n_1 + 2n_2 - 3, \eta(G) = 2n_1 + n_2 - 2, \mathcal{D}(G) = n_1 + n_2 - 2$$

Lemma 5.11.
We have $\mathcal{D}(G) \le \eta(G) \le s(G) - \exp(G) + 1$

Proof. Consider a sequence S of length $|S| \ge s(G) - \exp(G) + 1$, let $n = \exp(G)$. Now consider the sequence $T = 0^{n-1}S$ of length $|T| \ge s(G)$ then there is a subsequence $T' \mid T$ such that $\sigma(T') = 0$. $T' = 0^k S'$ where $0 \le k \le n-1$ and $S' \mid S$. Also note that $|T'| = |S'| + k = n$ and therefore S' is the required short zero sum subsequence of S. The other inequality holds by definition. $\square$

Lemma 5.12.
Let $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid n_2 \mid \cdots \mid n_r$. If $r \ge 2$ then $\eta(G) \ge \mathcal{D}^*(G) - 1 + n_1$

Proof. Let $G \cong < e_1 > \oplus \cdots \oplus < e_r >$ with $ord(e_i) = n_i$ Look at

$$e = \sum_{i=1}^{r} e_i, \quad S = e^{n_1 - 1} \prod_{i=1}^{r} e_i^{n_i - 1}$$

Enough to prove that S does not have any short zero-sum subsequence. Let

$$T = e^n \prod_{i=1}^{r} e_i^{n_i'}$$

with $0 \le n \le n_1 - 1$ and $0 \le n_i' \le n_i - 1$ for all $i = 1, 2, \ldots, r$ be a short zero sum subsequence of S. It is clear that $n \ge 1$ and thus we have $0 = \sigma(T) = (n_1' + n)e_1 + \cdots + (n_r' + n)e_r$ which implies that $n_i' + n \equiv 0 \pmod{n_i}$ for all $i = 1, 2, \ldots r$. We also note that $1 \le n_i' + n \le 2n_i - 2 \Rightarrow n_i' = n_i - n$ for all $i = 1, 2, \ldots, r$. Thus we have $|T| = n + \sum_{i=1}^{r}(n_i - n) > n_r = \exp(G)$ and we are done. $\square$

Lemma 5.13.
Let $H \subset G$ be a subgroup, $k \in \mathbb{N}$ and $\phi : G \to G/H$ be a group epimorphism. If S is a sequence over G and $|S| \ge (k-1)\exp(G/H) + s(G/H)$, then S admits a product decomposition $S = S_1 \cdot \cdots \cdot S_k S'$ where for every $i \in \{1, 2, \ldots, k\}$ $\phi(S_i)$ has a sum zero and length $|S_i| = \exp(G/H)$

Proof. Proof by induction. Suppose it is true for some $j \in \{1, 2, \ldots, k-1\}$ that is $S = S_1 S_2 \cdots S_j S'$ where $\phi(S_i)$ has zero sum for all $i = 1, 2, \ldots, j$ and $|S_i| = \exp(G/H)$. Then $|S'| = |S| - j\exp(G/H) \ge s(G/H)$ and thus S' also has a subsequence S_{j+1} such that $\phi(S_{j+1})$ has a zero sum and $|S_{j+1}| = \exp(G/H)$ and we are done. $\square$

Lemma 5.14.
Let $H \subset G$ be a subgroup. If S is a sequence over G and $|S| \ge (s(H)-1)\exp(G/H) + s(G/H)$, then S has a zero-sum subsequence T of length $|T| = \exp(H)\exp(G/H)$. In particular, if

$\exp(G) = \exp(H)\exp(G/H)$ then

$$s(G) \leq (s(H)-1)\exp(G/H) + s(G/H)$$

Proof. Let S be a sequence over G of length $|S| \geq (s(H)-1)\exp(G/H)+s(G/H)$. By lemma 5.13 we have the decomposition $S = S_1 S_2 \cdots S_{s(H)} S'$ where for every $i = 1, 2, \ldots, s(H)$, $\phi(S_i)$ has a zero sum and length $|S_i| = \exp(G/H)$. Now, consider the sequence $\sigma(S_1)\sigma(S_2)\cdots\sigma(S_{s(H)})$ over H, this has a zero sum subsequence of length $\exp(H)$ and thus we are done. □

Proof of theorem 5.10. $\exp(G) = n_2$. From lemma 5.12 and lemma 5.11 we have

$$\eta(G) \geq n_1 - 1 + n_2 - 1 + 1 - 1 + n_1 = 2n_1 + n_2 - 2$$

$$s(G) \geq \eta(G) + n_2 - 1 \geq 2n_1 + 2n_2 - 3$$

It suffices to prove that $s(G) \leq 2n_1 + 2n_2 - 3$.

We induct on $\exp(G)$. Let p be a prime such that $p|n_1 \Rightarrow n_1 = pm_1$ and $p|n_2 \Rightarrow n_2 = pm_2$.By the induction hypothesis the proposition is true for groups of the form $Q = C_{m_1} \oplus C_{m_2}$ and moreover $G/Q \cong C_p^2$.By lemma 5.14 and proposition 4.7 we have

$$s(G) \leq (s(Q)-1)\exp(G/Q) + s(G/Q) = (2m_1 + 2m_2 - 4)p + 4p - 3 = 2n_1 + 2n_2 - 3$$

And we are done. □

We have therefore found the values of $\mathcal{D}(G), s(G), \eta(G)$ for all finite abelian groups G of rank 2.

We have already obtained $s(G)$ and $\eta(G)$ for $G = C_2^r$ and

Now we will find values for $G = C_{2^k}^r$ in the usual way, finding the lower bound and then the upper bound.

Lower bound $n \geq 2, r \geq 1$

Lemma 5.15.

$$\eta(C_n^r) \geq (2^r - 1)(n-1) + 1$$
$$s(C_n^r) \geq 2^r(n-1) + 1$$

Proof. □

Upper bound $n, r, c \in \mathbb{N}$

Lemma 5.16.
If

$$\eta(C_m^r) \leq c(m-1) + 1 \text{ and}$$
$$\eta(C_n^r) \leq c(n-1) + 1$$

Then $\eta(C_{mn}^r) \le c(nm-1)+1$. Same result holds for $s(C_{nm}^r)$.

Proof. □

Question 5.17.
What about C_3^r ?

In C_3^r if we have $x+y+z=0 \Leftrightarrow x+z=-y \Leftrightarrow x+z=2y \Leftrightarrow x-y=y-z$. This means that x, y, z are a three term AP or are all equal.
Geometric Interpretation. Consider the elements of an AP to be $\{x, x+d, x+2d\} = \{x+td \mid t \in \mathbb{Z}/\mathbb{Z}_3\}$. Thus C_3^3 is the vector space over $\mathbb{Z}/\mathbb{Z}_3$. And three terms are in AP $\Leftrightarrow$ they form an affine line.

For the constant $s(G)$, we have the following two results

5.2.2 EGZ constant for groups of the form $C_2^r \oplus C_n$

Can be found in [FZ16], [She17]

6 Future reading prospects

- Inverse zero sum problems [Gir10] [GS20] [Pen+20] [Gao+08]
- Set based constants [BS10] [Ord+11] [GRT04] [Ord+09]
- Zero sums in semigroups [Wan20]
- Weighted zero sum constants [ZY11], [AC08] [ARS20]
- Zero sum problems in affine caps [Ede+06] [Ede08]
- Zero sum in abelian non-cyclic groups [Car95]
- More results on Davenport and EGZ constant [Chi+12] [Adh+06] [Alk08] [GHZ16] [Mor] [GG99]
- Zero sum subsequences of specified length [LW12] [Gao+14]
- Variant constants [Zak19b] [Tha07] [Per21]
- monoids, product one sequences, multiplicative ideal theory [CDG16] [GG13]

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